Free Quality School Education

Ministry of Basic and Senior Secondary Education

Pupils' Handbook for

Senior Secondary *Mathematics Revision*

Part II

STRICTLY NOT FOR SALE

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Please see final page for further information.

Foreword

These Lesson Plans and the accompanying Pupils' Handbooks are essential educational resources for the promotion of quality education in senior secondary schools in Sierra Leone. As Minister of Basic and Senior Secondary Education, I am pleased with the professional competencies demonstrated by the writers of these educational materials in English Language and Mathematics.

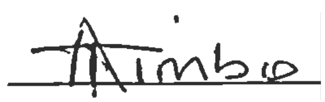
The Lesson Plans give teachers the support they need to cover each element of the national curriculum, as well as prepare pupils for the West African Examinations Council's (WAEC) examinations. The practice activities in the Pupils' Handbooks are designed to support self-study by pupils, and to give them additional opportunities to learn independently. In total, we have produced 516 lesson plans and 516 practice activities – one for each lesson, in each term, in each year, for each class. The production of these materials in a matter of months is a remarkable achievement.

These plans have been written by experienced Sierra Leoneans together with international educators. They have been reviewed by officials of my Ministry to ensure that they meet the specific needs of the Sierra Leonean population. They provide step-by-step guidance for each learning outcome, using a range of recognized techniques to deliver the best teaching.

I call on all teachers and heads of schools across the country to make the best use of these materials. We are supporting our teachers through a detailed training programme designed specifically for these new lesson plans. It is really important that the Lesson Plans and Pupils' Handbooks are used, together with any other materials they may have.

This is just the start of educational transformation in Sierra Leone as pronounced by His Excellency, the President of the Republic of Sierra Leone, Brigadier Rtd Julius Maada Bio. I am committed to continue to strive for the changes that will make our country stronger and better.

I do thank our partners for their continued support. Finally, I also thank the teachers of our country for their hard work in securing our future.



Mr. Alpha Osman Timbo

Minister of Basic and Senior Secondary Education

The policy of the Ministry of Basic and Senior Secondary Education, Sierra Leone, on textbooks stipulates that every printed book should have a lifespan of three years.

To achieve thus, DO NOT WRITE IN THE BOOKS.

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Introduction

to the Pupils' Handbook

These practice activities are aligned to the Lesson Plans, and are based on the National Curriculum and the West Africa Examination Council syllabus guidelines. They meet the requirements established by the Ministry of Basic and Senior Secondary Education.

1. The practice activities will not take the whole term, so use any extra time to revise material or re-do activities where you made mistakes.
2. Use other textbooks or resources to help you learn better and practise what you have learned in the lessons.
3. Read the questions carefully before answering them. After completing the practice activities, check your answers using the answer key at the end of the book.
4. Make sure you understand the learning outcomes for the practice activities and check to see that you have achieved them. Each lesson plan shows these using the symbol to the right.

   Learning Outcomes

5. Organise yourself so that you have enough time to complete all of the practice activities. If there is time, quickly revise what you learned in the lesson before starting the practice activities. If it is taking you too long to complete the activities, you may need more practice on that particular topic.
6. Seek help from your teacher or your peers if you are having trouble completing the practice activities independently.
7. Make sure you write the answers in your exercise book in a clear and systematic way so that your teacher can check your work and you can refer back to it when you prepare for examinations.
8. Congratulate yourself when you get questions right! Do not worry if you do not get the right answer – ask for help and continue practising!

KEY TAKEAWAYS FROM SIERRA LEONE'S PERFORMANCE IN WEST AFRICAN SENIOR SCHOOL CERTIFICATE EXAMINATION – GENERAL MATHEMATICS[1]

This section, seeks to outline key takeaways from assessing Sierra Leonean pupils' responses on the West African Senior School Certificate Examination. The common errors pupils make are highlighted below with the intention of giving teachers an insight into areas to focus on, to improve pupil performance on the examination. Suggestions are provided for addressing these issues.

Common errors

1. Errors in applying principles of BODMAS
2. Mistakes in simplifying fractions
3. Errors in application of Maths learned in class to real-life situations, and vis-a-versa.
4. Errors in solving geometric constructions.
5. Mistakes in solving problems on circle theorems.
6. Proofs are often left out from solutions, derivations are often missing from quadratic equations.

Suggested solutions

1. Practice answering questions to the detail requested
2. Practice re-reading questions to make sure all the components are answered.
3. If possible, procure as many geometry sets to practice geometry construction.
4. Check that depth and level of the lesson taught is appropriate for the grade level.

It is important that candidates understand what to expect on the day of the WASSCE exam. Details of the exam and strategies for taking the exam are given below.

[1] This information is derived from an evaluation of WAEC Examiner Reports, as well as input from WAEC examiners and Sierra Leonean teachers.

Content of the WASSCE Exam

The WASSCE Mathematics exam consists of 3 sections. These are described in detail below:

Paper 1 – Multiple Choice

- Paper 1 is 1.5 hours, and consists of 50 multiple choice questions. It is worth 50 marks.
- This gives 1.8 minutes per problem, so time must be planned accordingly.
- The questions are drawn from all topics on the WASSCE syllabus.

Paper 2 – Essay Questions

- Paper 2 consists of 13 essay questions in 2 sections – 2A and 2B.
- Paper 2 is worth 100 marks in total.
- Pupils will be required to answer 10 essay questions in all, across the 2 sections.
- This is an average of 15 minutes per question. However, keep in mind that section 2B is more complicated, so plan your time accordingly.

Paper 2A – Compulsory Questions

- Paper 2A is worth 40 marks.
- There are 5 **compulsory** essay questions in paper 2A.
- Compulsory questions often have multiple parts (a, b, c, …). The questions may not be related to each other. Each part of the question should be completed.
- The questions in paper 2A are simpler than those in 2B, generally requiring fewer steps.
- The questions in paper 2A are drawn from the common area of the WASSCE syllabus (they are not on topics that are specific to certain countries).

Paper 2B – Advanced Questions

- Paper 2B is worth 60 marks.
- There are 8 essay questions in paper 2A, and candidates are expected to answer 5 of them.
- Questions in section 2B are of a greater length and difficulty than section 2A.
- A maximum of 2 questions (from among the 8) may be drawn from parts of the WASSCE syllabus that are not meant for Sierra Leone. These topics are not in the Sierra Leone national curriculum for secondary schools. Candidates from Sierra Leone may choose to answer such questions, but it is not required.
- Choose 5 questions on topics that you are more comfortable with.

Exam Day

- Candidates should bring a pencil, geometry set, and scientific calculator to the WASSCE exam.
- Candidates are allowed to use log books (logarithm and trigonometry tables), which are provided in the exam room.

Exam-taking skills and strategies

- Candidates should read and follow the instructions carefully. For example, it may be stated that a trigonometry table should be used. In this case, it is important that a table is used and not a calculator.
- Plan your time. Do not spend too much time on one problem.
- For essay questions, show all of your working on the exam paper. Examiners can give some credit for rough working. Do not cross out working.
- If you complete the exam, take the time to check your solutions. If you notice an incorrect answer, double check it before changing it.
- For section 2B, it is a good idea to skim read the questions quickly before getting started. If you see a topic that you are familiar and comfortable with, work on those problems. Try not to spend a lot of time deciding which problems to solve, or thinking about problems you will not solve.

Lesson Title: Measuring angles	**Theme:** Geometry
Practice Activity: PHM4-L049	**Class:** SSS 4

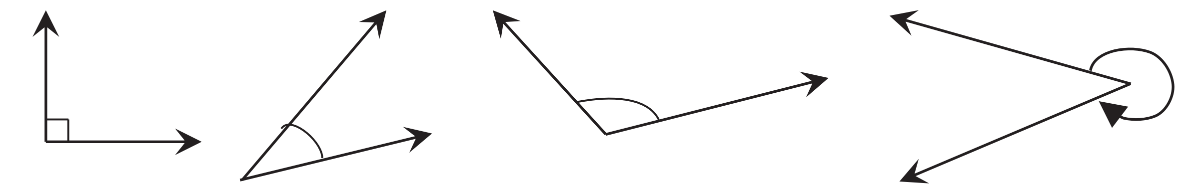
Learning Outcomes

By the end of the lesson, you will be able to:

1. Identify various types of angles (acute, obtuse, right, reflex, straight).
2. Measure angles using a protractor.

Overview

There are 4 main categories of angles that you should be familiar with. These are:



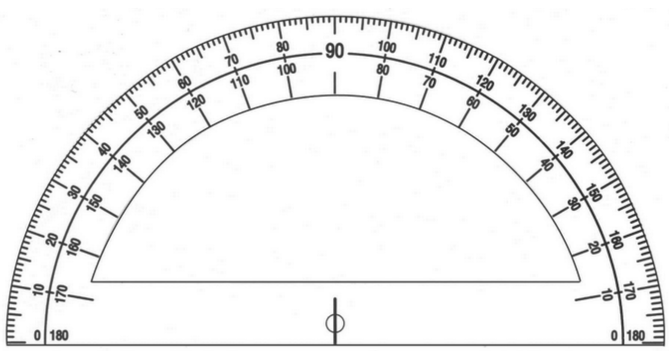
Right angle
90°

Acute angle
less than 90°

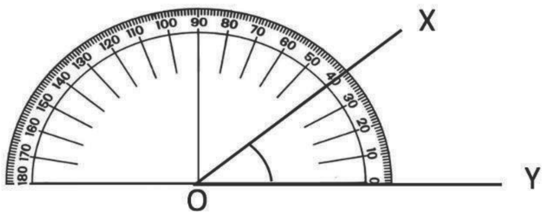
Obtuse angle
greater than 90° and less than 180°

Reflex angle
greater than 180° and less than 360°

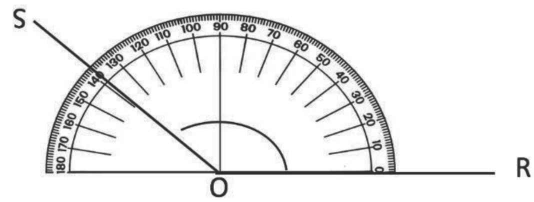
Recall that a full rotation is 360°. The tool used to measure angles is called a protractor. If you have a protractor, use it throughout this lesson. If you do not have a protractor, make your own. You can trace the following diagram onto a piece of paper and cut it out:



To measure an angle, the centre of the straight side of the protractor should be placed at the vertex. The straight line of the protractor is along one side of the angle. The other side of the angle gives its measure. For example:



Acute angle
$\angle XOY = 40°$

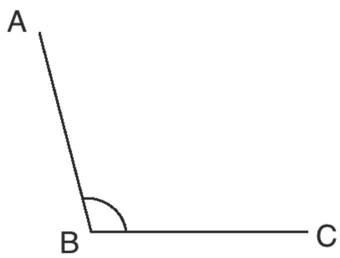


Obtuse angle
$\angle SOR = 139°$

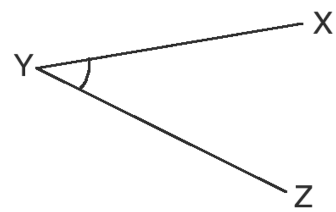
Solved Examples

1. Measure each of the following angles with a protractor:

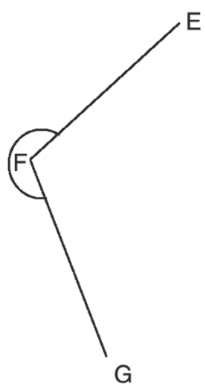
a.



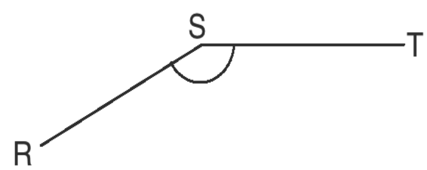
b.



c.



d.

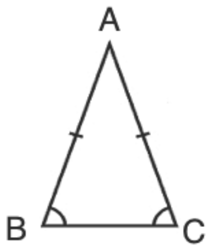


Solutions:

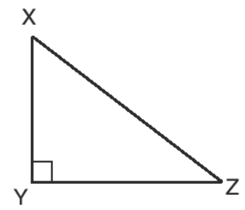
a. $\angle ABC = 105°$
b. $\angle XYZ = 36°$
c. $\angle EFG = 248°$
d. $\angle RST = 148°$

2. Determine whether each interior angle of the following shapes is acute, obtuse, right, or reflex. Recall that interior angles are the angles inside the shapes.

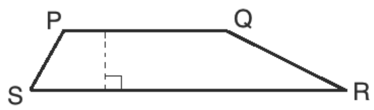
a.



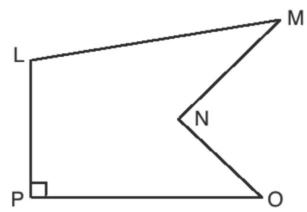
b.



c.



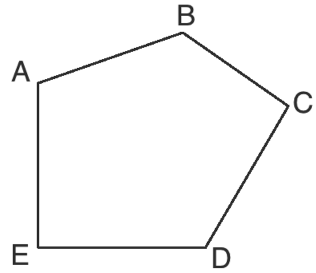
d.



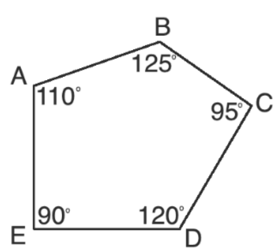
Solutions:
Measure each angle if needed. However, you can generally categorise angles by looking at them.

a. A, B and C are acute.
b. Y is a right angle; X and Y are acute.
c. S and R are acute; P and Q are obtuse.
d. P is a right angle; L is an obtuse angle; M and O are acute angles; N is a reflex angle.

3. Measure and label each interior angle of the pentagon below. What is the sum of the angles?

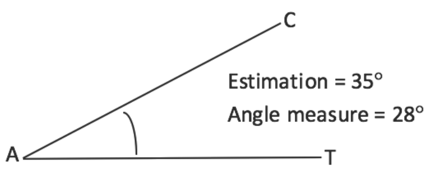


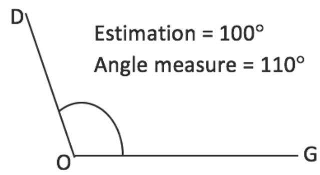
Solution:
Measure each angle to the nearest degree with a protractor, and label them as follows:



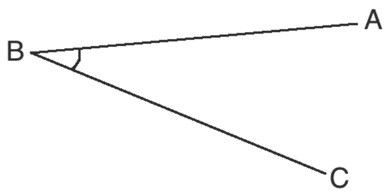
Add the angles to find their sum: $90° + 110° + 125° + 95° + 120° = 540°$
Note that the sum of the angles of a pentagon is always $540°$.

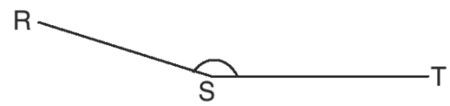
4. Draw an acute angle and an obtuse angle. Estimate the measure of each. Then measure each angle with a protractor.
   Solution:
   The solution will depend on the angle you draw. Example solutions are:





Practice

1. Measure the following angles with a protractor:
   a.


   b.


2. Draw shapes with the following characteristics:
   a. A triangle with a $70°$ angle.
   b. A quadrilateral with a $90°$ angle.
   c. A pentagon with exactly 2 acute angles.
3. Categorise the following angles as acute, obtuse, or reflex:
   a. $91°$
   b. $210°$
   c. $9°$
   d. $345°$
   e. $102°$
   f. $29°$
   g. $170°$

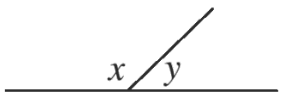
Lesson Title: Solving for angles – Part 1	**Theme:** Geometry
Practice Activity: PHM4-L050	**Class:** SSS 4

Learning Outcome
By the end of the lesson, you will be able to solve for angles given intersecting lines, including parallel lines with a transversal.

Overview

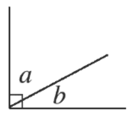
This lesson is on solving for angles when there are intersecting lines. Recall the following related definitions:

Supplementary angles:



Sum to 180°.
In the diagram, $x + y = 180°$

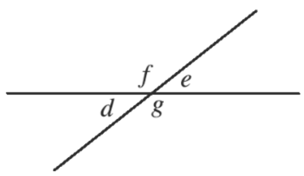
Complementary angles:



Sum to 90°.
In the diagram, $a + b = 90°$

These definitions can often be used to solve for missing angles in diagrams.

There are also rules that we can use related to the intersection of 2 lines. Consider the diagram, which shows 2 intersecting lines:

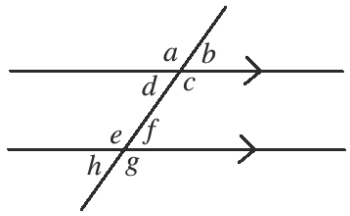


When 2 lines intersect, the opposite angles are equal. In the diagram above, $d = e$ and $f = g$.

Also note that there are supplementary angles in the diagram. From the diagram, we have the following:

- $f + e = 180°$
- $e + g = 180°$
- $d + g = 180°$
- $d + f = 180°$

There are also rules for parallel lines with a transversal. A transversal is a line that intersects both parallel lines. Consider the following diagram:

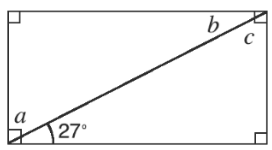


The rules given above apply. For example, a and b are supplementary angles, and sum to $180°$. a and c are opposite angles, and are equal. Also note that there is a relationship between the angles of the 2 intersections.

- **Alternate angles** are **equal**. These are angles on opposite sides of the transversal, inside of the parallel lines. In the diagram, alternate angles are: d and f; c and e.
- **Corresponding angles** are **equal**. These are angles that are in the same position in the 2 intersections. In the diagram, corresponding angles are: a and e; b and f; c and g; d and h.
- **Co-interior angles** are **supplementary**. These are angle on the same side of the transversal line, inside of the parallel lines. In the diagram, co-interior angles are: c and f; d and e.

Solved Examples

1. In the diagram, find the measures of a, b and c:

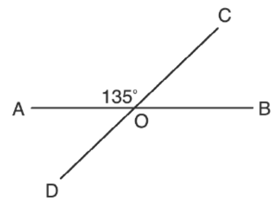


Solution:
Note that a and $27°$ form a right angle. In other words, they are complementary. Subtract from $90°$ to find the measure of a: $90° - 27° = 63°$.

There are 2 ways to solve for b and c. Note that the diagonal line is a transversal of two parallel sides of the rectangle. Thus, $b = 27°$ because these are alternate interior angles. Likewise, c and a are alternate interior angles, and $c = a = 63°$.

Alternatively, you may also use the right triangles on either side of the diagonal line to solve for b and c. Recall that the angles in a triangle sum to $180°$.

2. In the diagram below, find the measures of: a. $\angle COB$ b. $\angle BOD$ c. $\angle AOD$



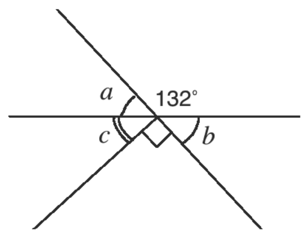
Solutions:

a. Note that $\angle AOC$ and $\angle COB$ form a straight line, and thus are supplementary angles. Subtract from $180°$ to find $\angle COB$: $\angle COB = 180° - 135° = 45°$

b. Note that $\angle AOC$ and $\angle BOD$ are opposite angles, and are thus equal. $\angle BOD = \angle AOC = 135°$

c. Angle $\angle AOD$ can be found in multiple ways. Use the opposite angle $\angle COB$, or the supplementary angles $\angle AOC$ or $\angle BOD$. We have $\angle AOD = \angle COB = 45°$.

3. In the diagram below, find the measures of a, b and c:

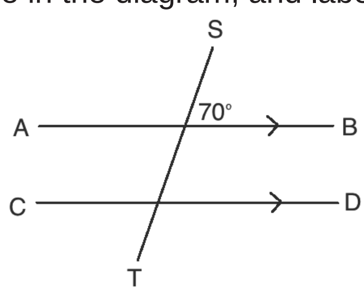


Solution:

Note that the following angles are supplementary: a and $132°$, and b and $132°$. Thus, $a = b = 180° - 132° = 48°$.

There are 2 ways to find c. One way is to subtract all of the known angles from $360°$ (one full rotation). Alternatively, subtract $90°$ and b from $180°$. These angles form a straight line ($180°$). Thus, $c = 180° - 90° - 48° = 42°$.

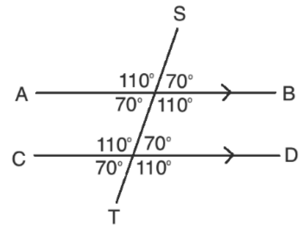
4. Find the measure of each angle in the diagram, and label them:



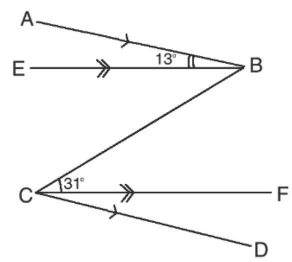
Solution:

Note that the following angles are equal to the labeled angle, $70°$: the opposite angle, and corresponding angles in the other intersection.

Find the angles that are supplementary to the labeled angle by subtracting from $180°$: $180° - 70° = 110°$. Label the supplementary angles $110°$. Note that the corresponding angles in the other intersection are also $110°$.

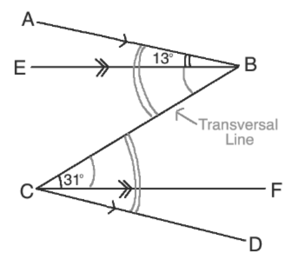


5. In the diagram below, $AB||CD$ and $EB||CF$. Find the measure of $\angle BCD$.



Solution:

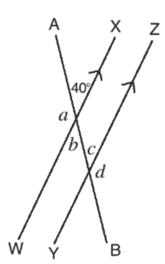
Note that BC is a transversal for both sets of parallel lines. Thus, we can label the alternate angles as equal. We have $\angle ABC = \angle BCD$ and ∠EBC=∠BCF. This is shown in the diagram:



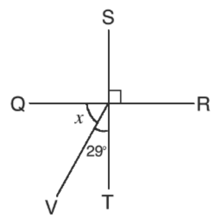
Since we are given $\angle BCF = 31°$, we also have $\angle EBC = 31°$. Add $\angle ABE$ and $\angle EBC$ to find $\angle ABC$: $\angle ABC = \angle ABE + \angle EBC = 13° + 31° = 44°$.
Therefore, $\angle BCD = \angle ABC = 44°$

Practice

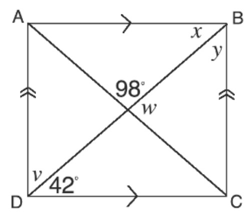
1. In the diagram below, find the measures of a, b, c and d:



2. Find the measure of x:



3. In the diagram, ABCD is a rectangle. Find the measures of $\boldsymbol{v}$, w, x and y:

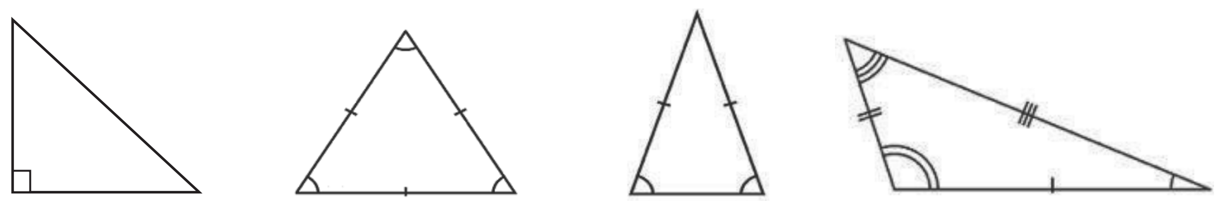


Lesson Title: Solving for angles – Part 2	Theme: Geometry
Practice Activity: PHM4-L051	**Class:** SSS 4

Learning Outcome
By the end of the lesson, you will be able to solve for angles in triangles.

Overview

Recall the following types of triangles and their properties:



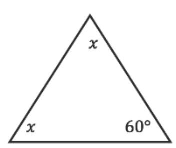
Right-angled	**Equilateral**	**Isosceles**	**Scalene**
one right angle	3 equal sides 3 equal angles, 60°	2 equal sides 2 equal angles	no equal sides or angles

The angles of a triangle always add up to 180 degrees. To find unknown angles, subtract the known angles from 180°.

You may also use the properties of triangles to solve them. For example, if you know any 1 angle in an isosceles triangle you can solve for the other 2, using the fact that 2 angles are equal (see solved example 3).

Solved Examples

1. Find the measure of angle x in the figure below:

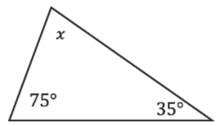


Solution:

$$\begin{aligned}
x + x + 60^0 &= 180^o && \text{Sum of angles is } 180^o \\
2x + 60^o &= 180^o \\
2x &= 180^o - 60^o && \text{Transpose } 60^o \\
2x &= 120^o \\
\frac{2x}{2} &= \frac{120^o}{2} && \text{Divide throughout by 2} \\
x &= 60^o
\end{aligned}$$

You may also note that the triangle is an equilateral triangle. Equilateral triangles have all of their angles equal to 60^{o}.

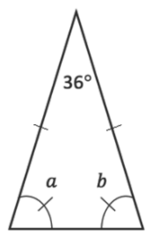
2. Find the value of the angle in the diagram below.



Solution:

$$\begin{aligned} 75° + 35^{o} + x^{o} &= 180^{o} && \text{Sum of angles is } 180^{o} \\ 110^{o} + x^{o} &= 180^{o} \\ x &= 180^{o} - 110^{o} && \text{Transpose } 100° \\ x &= 70^{o} \end{aligned}$$

3. Find the measures of a and b:



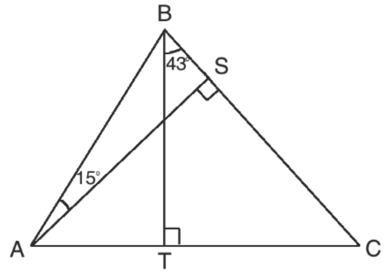
Solution:

Subtract the known angle from 180: $180° - 36° = 144°$.

We know that $a + b = 144°$, and that a and b are equal.

Divide $144°$ by 2 to find the measure of both a and b: $a = b = 144° \div 2 = 72°$

4. In the triangle below, AS and BT are 2 altitudes of $\angle ABC$. $\angle CBT = 43°$ and $\angle BAS = 15°$. Find the measures of the following angles:
   a. $\angle ACS$ b. $\angle CAS$ c. $\angle ABC$



Solution:

a. To find $\angle ACS$, use ΔBCT. Two angles are known, $90°$ and $43°$.

$$\begin{aligned} 90° + 43° + \angle ACS &= 180^{o} && \text{Sum of the angles is } 180^{o} \\ 133° + \angle ACS &= 180^{o} \\ \angle ACS &= 180^{o} - 133° && \text{Transpose } 133^{o} \\ \angle ACS &= 47° \end{aligned}$$

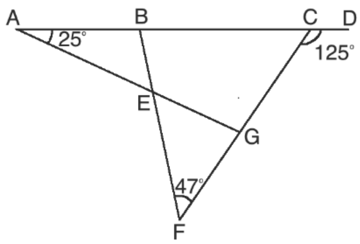
b. To find $\angle CAS$, use ΔCAS. Two angles are known, $90°$ and $47°$.

$$\begin{aligned} 90° + 47° + \angle CAS &= 180^o && \text{Sum of the angles is } 180^o \\ 137° + \angle CAS &= 180^o \\ \angle CAS &= 180^o - 137° && \text{Transpose } 137^o \\ \angle CAS &= 43° \end{aligned}$$

c. To find $\angle ABC$, use ΔABS. Two angles are known, $90°$ and $15°$.

$$\begin{aligned} 90° + 15° + \angle ABC &= 180^o && \text{Sum of the angles is } 180^o \\ 105° + \angle ABC &= 180^o \\ \angle ABC &= 180^o - 105° && \text{Transpose } 105^o \\ \angle ABC &= 75° \end{aligned}$$

5. In the diagram below, $ABCD$ is a straight line. $\angle BAE = 25°$, $\angle EFG = 47°$, and $\angle DCG = 125°$. Find $\angle BE$.



Solution:
This problem can be solved by using various triangles in the diagram. Notice that there are several triangles: ΔABE, ΔEFG, ΔACG, ΔBCF. Also use what you know of the supplementary angles.

Start to solve for angles in the diagram. As you solve for angles, label them (see labeled diagram below).

Find $\angle BCF$: It is supplementary to $\angle DCG$, so subtract from $180°$:

$$\angle BCF = 180° - 125° = 55°$$

Find $\angle CBF$: Use ΔBCF, in which there are 2 known angles; subtract from $180°$:

$$\angle CBF = 180° - 55° - 47° = 78°$$

Find $\angle ABE$: It is supplementary to $\angle CBF$, so subtract from $180°$:

$$\angle ABE = 180° - 78° = 102°$$

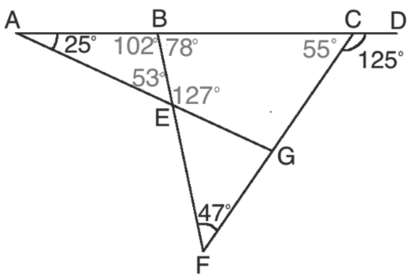
Find $\angle AEB$: Use ΔABE, in which there are 2 known angles; subtract from $180°$:

$$\angle AEB = 180° - 25° - 102° = 53°$$

Find $\angle BEG$: It is supplementary to $\angle AEB$, so subtract from $180°$:

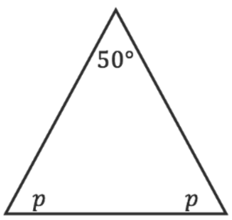
$$\angle BEG = 180° - 53° = 127°$$

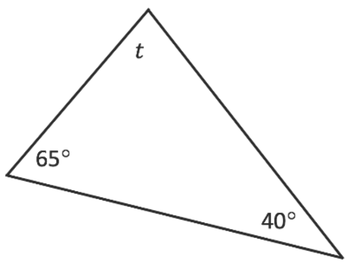
Answer: $\angle BEG = 127°$

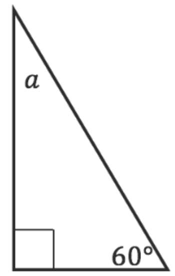


Practice

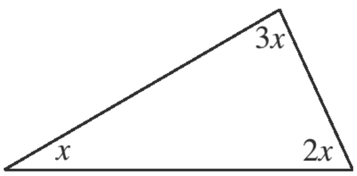
1. Find the values of the variables in the diagrams below:

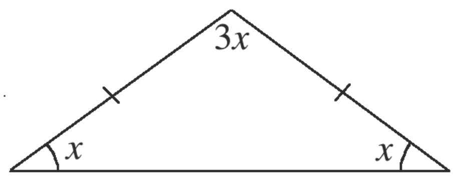
a. 

b. 

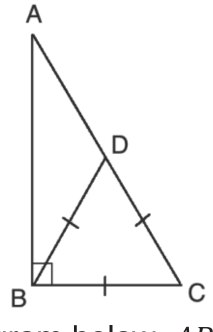
c. 

2. Find the value of x in each of the diagrams below:

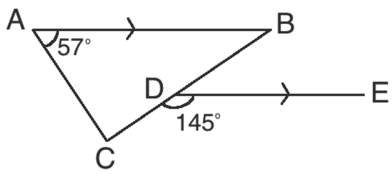
a. 

b. 

3. In the diagram below, ΔABC is a right-angled triangle and $|BD| = |CD| = |BC|$. Find the measure of $\angle BAC$.



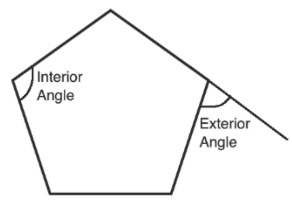
4. In the diagram below, $AB||DE$, $\angle BAC = 57°$, and $\angle CDE = 145°$. Find $\angle ACB$.



Lesson Title: Solving for angles – Part 3	**Theme:** Geometry
Practice Activity: PHM4-L052	**Class:** SSS 4

Learning Outcome
By the end of the lesson, you will be able to solve for angles in quadrilaterals and other polygons.

Overview



The properties of the interior and exterior angles of polygons can be used to solve problems. **Interior angles** are inside of a polygon, and **exterior angles** are formed on the outside by extending the sides of the polygon. Every polygon has the same number of interior and exterior angles as it has sides.

This is the formula for finding the **sum** of the interior angles in a polygon:

$(n-2) \times 180°$ where n is the number of sides.

The sum of interior angles up to a decagon are given in the table below. It is a good idea to memorise these, as well as understand how to apply the above formula.

Sides	Name	Sum of Interior Angles
3	Triangle	180°
4	Quadrilateral	360°
5	Pentagon	540°
6	Hexagon	720°
7	Heptagon	900°
8	Octagon	1,080°
9	Nonagon	1,260°
10	Decagon	1,440°

For regular polygons, all of the interior angles are equal. There is a formula for finding the measure of interior angles of a **regular** polygon: $\frac{(n-2)\times 180°}{n}$ where n is the number of sides.

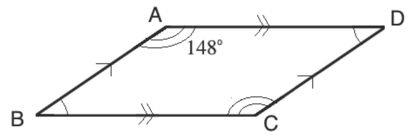
For polygons that are not regular, missing angles can be found by subtracting known angles from the sum of the angles for that type of polygon.

Exterior angles can be found if you know the corresponding interior angle. Each exterior angle and the adjacent interior angle are supplementary, and sum to 180°.

Exterior angles can also be found if you are given known exterior angles. All of the exterior angles of **any** polygon sum to 360°. The exterior angles of a regular polygon are all equal. We can use the following formula to find the measure of each exterior angle: $\frac{360°}{n}$ where n is the number of sides.

Solved Examples

1. Find the measures of angles B, C, and D in the parallelogram:



Solution:

Note that $C = A = 148°$

$$\begin{aligned} A + B + C + D &= 360° \\ 148° + B + 148° + D &= 360° \\ B + D + 296° &= 360° \\ B + D &= 360° - 296° \\ B + D &= 64° \end{aligned}$$

Since $B = D$, divide $64°$ by 2 to find the measure of each:
$B = D = 64° \div 2 = 32°$

2. Quadrilateral $WXYZ$ has four angles which are in the ratio $2:4:4:5$. Find the degree measure of the smallest angle of quadrilateral $WXYZ$.

Solution:

Method 1. The smallest angle is the one given by 2 in the ratio. We know that the sum of interior angles of a quadrilateral is $360°$. Multiply this total by $\frac{2}{15}$, the ratio of the smallest angle:

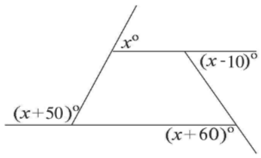
Smallest angle $= \frac{2}{15} \times 360° = 48°$

Method 2. Let x be a common factor of the 4 angles. Then the sum of the angles is $2x + 4x + 4x + 5x$. Set this equal to $360°$ and solve for x:

$$\begin{aligned} 2x + 4x + 4x + 5x &= 360° \\ 15x &= 360° \\ x &= \frac{360°}{15} = 24° \end{aligned}$$

$2x$ represents the smallest angle. Multiply: $2x = 2(24) = 48°$

3. The exterior angles of the quadrilateral below are x^{o}, $(x - 10)^{o}$, $(x + 60)^{o}$ and $(x + 50)^{o}$. Find the value of x.



Solution:

Set the sum of the angles equal to $360°$ and solve for x:

$$
\begin{aligned}
x^{\circ} + (x^{\circ} - 10)^{\circ} + (x + 60)^{\circ} + (x + 50)^{\circ} &= 360° \\
4x + 100^{\circ} &= 360° \\
4x &= 360° - 100^{\circ} \\
4x &= 260^{\circ} \\
\frac{4x}{4} &= \frac{260^{\circ}}{4} \\
x &= 65^{\circ}
\end{aligned}
$$

4. Calculate the sum of the interior angles of a polygon with 15 sides.

Solution:

Substitute $n = 15$ in the formula and solve:

$$
\begin{aligned}
\text{Sum of angles} &= (n-2) \times 180° \\
&= (15-2) \times 180° \\
&= 13 \times 180° \\
&= 2{,}340°
\end{aligned}
$$

5. The sum of the interior angles of a regular polygon is $900°$. How many sides does it have?

Solution:

Sum of angles	$=$	$(n-2) \times 180°$	
$900°$	$=$	$(n-2) \times 180°$	Substitute the values
$900°$	$=$	$180°n - 360°$	Clear bracket
$900° + 360°$	$=$	$180°n$	Transpose $-360°$
$1260°$	$=$	$180°n$	Divide throughout by $180°$
$\frac{1260°}{180°}$	$=$	n	Simplify
n	$=$	7	

The polygon has 7 sides. It is a heptagon.

6. The interior angle of a regular polygon is $108°$. Find the number of sides of the polygon.

Solution:

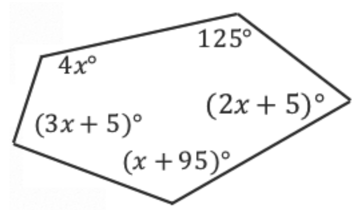
Apply the formula for interior angle, and solve for n:

$$108° = \frac{(n-2)\times 180°}{n}$$

$$
\begin{aligned}
108°n &= (n-2)\times 180° \\
108°n &= 180°n - 360° \\
108°n - 180°n &= -360° \\
-72°n &= -360° \\
\frac{-72°n}{-72°} &= \frac{-360°}{-72°} \\
n &= 5
\end{aligned}
$$

The polygon has 5 sides. It is a pentagon.

7. The interior angles are given in the pentagon at right. Solve for x:



Solution:

The angles are labeled in this pentagon but there is an unknown variable. Recall that the angles of a pentagon add up to 540°. We can add the interior angles and set them equal to 540°. Then, solve for x.

$540°$	$=$	$125° + (2x+5)° + (x+95)° + (3x+5)° + 4x°$	Add the angles
	$=$	$(125° + 5° + 95° + 5°) + (2x + x + 3x + 4x)°$	Combine like terms
	$=$	$230° + 10x°$	
$540° - 230°$	$=$	$10x°$	
$310°$	$=$	$10x°$	Transpose $230°$
$31°$	$=$	x	Divide by $10°$

8. A regular polygon has exterior angles measuring 40°. How many sides does the polygon have?

Solution:

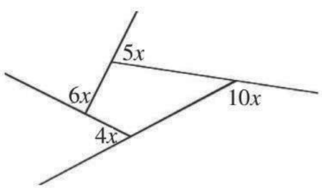
Exterior angle	$=$	$\frac{360°}{n}$	
$40°$	$=$	$\frac{360°}{n}$	Substitute $40°$
$40° \times n$	$=$	$360°$	Multiply throughout by n
n	$=$	$\frac{360°}{40°}$	Divide throughout by $40°$
n	$=$	9	

The polygon has 9 sides. It is a nonagon.

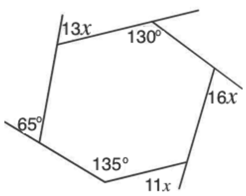
Practice

1. Find the value of x in each of the diagrams below:

a.



b.



2. The sum of the interior angles of a polygon is $4{,}680°$. How many sides does it have?
3. Each interior angle of a regular polygon is $140°$. Find the number of sides of the polygon.
4. A regular polygon has 15 sides. Find using the formulae:
    a. The measure of each exterior angle.
    b. The measure of each interior angle.
    c. The sum of the interior angles.
5. A heptagon has the following interior angles: $x°, (x + 4)°, (x + 8)°, (x + 14)°, (x + 26)°, (x + 34)°$, and $(x + 51)°$. Find the value of x.

Lesson Title: Solving for angles – Part 4	**Theme:** Geometry
Practice Activity: PHM4-L053	**Class:** SSS 4

Learning Outcome
By the end of the lesson, you will be able to solve for angles in compound and complex shapes.

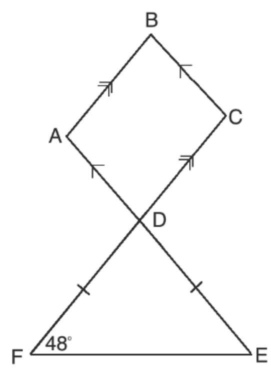
Overview

This lesson applies the information on solving angles from the previous lessons. You will encounter compound and complex shapes, and you will apply various rule and theorems to solve them.

When you encounter a complex shape, break it down to its parts. For example, you may find a triangle within a larger shape, and you may be able to solve for an angle of that triangle. Look for a strategy for finding the angle the problem asks you to solve.

Solved Examples

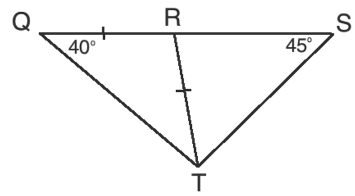
1. In the given shape, ΔFED is an isosceles triangle. $AB||DC$, and $BC||AE$. Find:
   a. $\angle ADF$
   b. $\angle BAD$
   c. $\angle BCD$



Solutions:

a. Note that $\angle ADF$ is supplementary to an angle in ΔDEF. To find $\angle ADF$, first solve ΔDEF. Since ΔDEF is isosceles, $\angle DFE = \angle DEF = 48°$.
   Therefore, $\angle EDF = 180° - 2(48°) = 84°$
   Using the fact that $\angle ADF$ and $\angle EDF$ are supplementary, we have:
   $$\angle ADF = 180° - \angle EDF = 180° - 84° = 96°$$
b. Note that $\angle BAD$ and $\angle ADF$ are alternate angles, which means they are equal. Therefore, $\angle BAD = \angle BAD = 96°$
c. Note that the opposite angles in a parallelogram are equal. Therefore, $\angle BCD = \angle BAD = 96°$.

2. In the diagram, QS is a straight line, $|QR| = |RT|$, $\angle RQT = 40°$ and $\angle RST = 45°$. Find $\angle RTS$.



Solution:
First, solve ΔQRT. This will give information that can be used to solve ΔQRT and find $\angle RTS$.

Step 1. Solve ΔQRT:

Because ΔQRT is isosceles, $\angle RQT = \angle QTR = 40°$

Calculate $\angle QRT$: $\angle QRT = 180° - 2(40°) = 100°$

Step 2. Solve ΔRST:

Use the fact that $\angle QRT$ and $\angle SRT$ are supplementary:

$$\angle SRT = 180° - \angle QRT = 180° - 100° = 80°$$

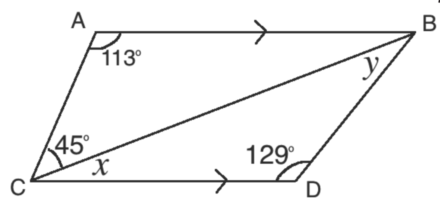
Subtract to find $\angle RTS$:

$$\angle RTS = 180° - 80° - 45° = 55°$$

Answer: $\angle RTS = 55°$

Note that this problem could also be solved by finding the measure of $\angle QTS$ of the larger triangle, and subtracting the measure of $\angle QTR$. There are often multiple ways to solve angle problems. Use the method that works best for you.

3. In the diagram below, $AB||CD$. Find the measures of x and y.



Solution:
Note that $\angle BAC$ and $\angle ACD$ are co-interior angles, which means they are supplementary. Solve for $\angle ACD$:

$$\angle ACD = 180° - \angle BAC = 180° - 113° = 67°$$

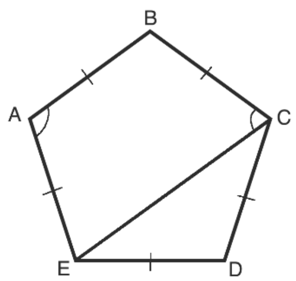
Subtract $45°$ from $\angle ACD$ to find x:

$$x = \angle ACD - 45° = 67° - 45° = 22°$$

Solve ΔBCD to find the measure of y:

$$y = 180° - 129° - x = 180° - 129° - 22° = 29°$$

4. In the diagram below, $ABCDE$ is a regular polygon. Find the measure of angles $\angle BAE$ and $\angle BCE$.



Solution:

To find $\angle BAE$, apply the formula for the interior angle of a regular polygon:

$$\angle BAE = \frac{(n-2)\times 180°}{n} = \frac{(5-2)\times 180°}{5} = \frac{3\times 180°}{5} = 108°$$

Note that ΔCDE is an isosceles triangle, and $\angle CDE = 108°$ because it is an angle of the regular pentagon. Also note that $\angle BCE$ can be solved for by finding $\angle DCE$ and subtracting it from $\angle BCD$, which is also $108°$.

Solve ΔCDE to find $\angle DCE$:

Subtract $\angle CDE$ from $180°$:

$$180° - \angle CDE = 180° - 108° = 72°$$

Divide by 2 to find $\angle DCE$:

$$72° \div 2 = 36°$$

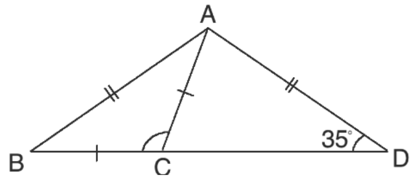
Subtract $\angle DCE$ from $\angle BCD$ to find $\angle BCE$:

$$\angle BCE = \angle BCD - \angle DCE = 108° - 36° = 72°$$

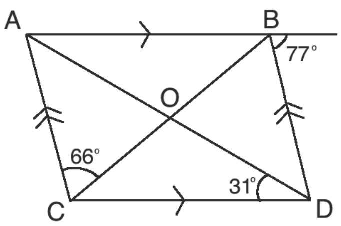
Answer: $\angle BAE = 108°$ and $\angle BCE° = 72°$

Practice

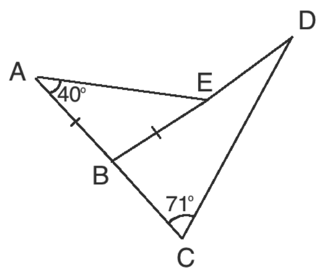
1. In the diagram below, |AB|=|AD| and |AC|=|BC|. $\angle ADC = 35°$. Find the size of:
   a. $\angle ABD$ b.$\angle ACB$ c. $\angle CAD$



2. In the diagram below, ABCD is a parallelogram. Its diagonals intersect at point O. Find the measures of the following angles:
   a. $\angle OCD$
   b. $\angle AOB$



3. In the diagram below, ABE is an isosceles triangle. $|AB| = |BE|$, $\angle BAE = 40°$, $\angle BCD = 71°$ and ABC is a straight line. Find the measure of:
   a. $\angle AEB$
   b. $\angle ABE$
   c. $\angle DBC$
   d. $\angle BDC$



Lesson Title: Angle problem solving	**Theme:** Geometry
Practice Activity: PHM4-L054	**Class:** SSS 4

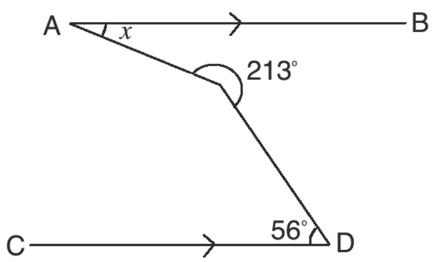
Learning Outcome
By the end of the lesson, you will be able to apply angle theorems and properties to solve problems.

Overview

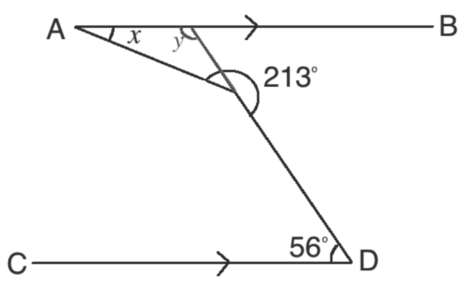
This lesson is practice on solving problems involving angles. You may encounter various types of problems on the WASSCE exam that require you to solve for missing angles. These may take the form of compound shapes, or they may just be a set of lines. Apply the rules and theorems from the previous lessons, and your problem-solving skills.

Solved Examples

1. In the diagram below, $AB||CD$. Find the measure of angle x.



Solution:
Note that the line from D can be extended to become a transversal of the parallel lines (see below). This makes a triangle with interior angle x that can be solved.



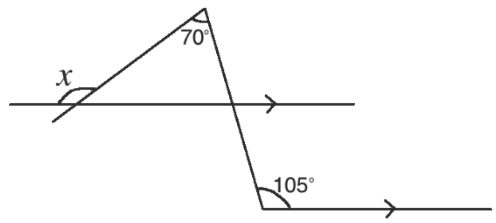
The newly formed angle y in the diagram is a co-interior angle with $56°$, which means they are supplementary. Therefore, $y = 180° - 56° = 124°$

Note that the other interior angle of the triangle can be found by subtracting $180°$ (the straight line) from $213°$ (the given angle): $213° - 180° = 33°$

Now that 2 interior angles of the triangle are known, subtract from $180°$ to find x:

$$x = 180° - 124° - 33° = 23°$$

2. Find the measure of angle x in the diagram below:



Solution:

Note that $105°$ has a corresponding angle above it, which is also supplementary to an interior angle of the triangle. Label it as $105°$ (see the labeled diagram below).

Solve for the interior angle of the triangle labeled a in the diagram below:

$a = 180° - 105° = 75°$

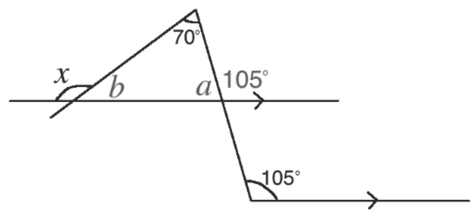
Solve for the other interior angle of the triangle, labeled b in the diagram below:

$\mathrm{b} = 180° - 70° - 75° = 35°$

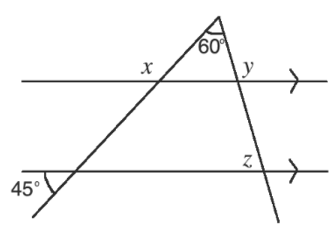
Solve for x, using the fact that it is supplementary to b:

$x = 180° - 35° = 145°$

Labeled diagram:



3. Solve for x, y and z in the diagram below.



Solution:

As you solve the angles, label them in the diagram (see below).

Note that there is an angle which corresponds to $45°$, and is also supplementary to x. Thus, subtract to find x: $x = 180° - 45° = 135°$

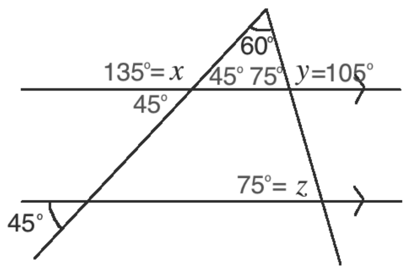
4. The small triangle that contains $60°$ can now be solved, which will allow us to find y. The angle supplementary to x is $180° - 135° = 45°$. The other angle in the triangle

can be found by subtracting: $180° - 60° - 45° = 75°$. Angle y is supplementary to this angle, which gives $y = 180° - 75° = 105°$.

Note z corresponds to the $75°$ interior angle of the small triangle. Thus, $z = 75°$.

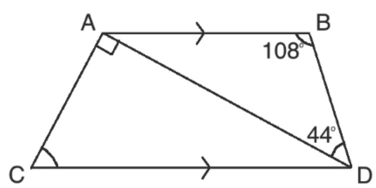
Answer: $x = 135°$, $y = 105°$, $z = 75°$.

Labeled diagram:

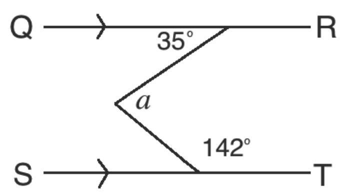


Practice

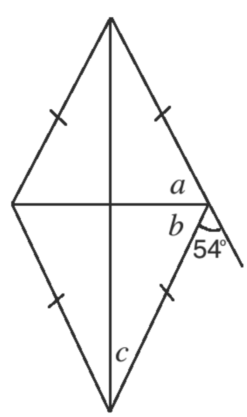
1. In the diagram below, $AB||CD$, $\angle ABD = 108°$ and $\angle ADB = 44°$. $\angle CAD$ is a right angle. Find the measure of: i. $\angle BAD$ ii. $\angle ADC$ iii. $\angle ACD$



2. In the diagram below, $QR||ST$. Find the measure of angle a.



3. In the diagram below, find angles a, b and c:



Lesson Title: Conversion of units of measurement	**Theme:** Mensuration
Practice Activity: PHM4-L055	**Class:** SSS 4

Learning Outcomes
By the end of the lesson, you will be able to:
1. Convert from large units to smaller units of measurement.
2. Convert from smaller units to larger units of measurement.

Overview

This lesson is on converting various common units of measurement. To convert from a smaller unit to a larger unit, **divide** by the conversion factor. To convert from a larger unit to a smaller unit, **multiply** by the conversion factor. Some common conversion factors are listed below.

Length / Distance		
10 decimetres	=	1 metre
100 centimetres	=	1 metre
1,000 millimetres	=	1 metre
1 kilometre	=	1,000 metres

Mass / Weight		
1 gramme	=	1,000 milligrammes
1 kilogrammes	=	1,000 grammes
1 ton	=	1,000 kilogrammes

Volume / Capacity		
1 litre	=	1,000 millilitres
1 litre	=	1,000 cm^3

It is important to note that units of measurement must be the same for 2 quantities to be added or subtracted from one another. For example, millilitres cannot be subtracted from litres. They must first be converted to the same measurement before the operation is applied.

You may apply more than 1 conversion factor to solve a problem. For example, to convert kilometres to centimetres you may first convert kilometres to metres, then convert metres to centimetres.

You may be asked to convert square or cubic measurements. For example, consider 1 m^3, which is a measurement of capacity. To convert this to cm^3, you must apply the conversion factor three times, which is the conversion factor to the 3rd power:

$$1\text{ m}^3 = 1\text{ m} \times 1\text{m} \times 1\text{m} = 100\text{ cm} \times 100\text{ cm} \times 100\text{ cm} = 1{,}000{,}000\text{ cm}^3$$

Solved Examples

1. Convert the following as indicated:
   a. 5 kilometres to metres
   b. $2\frac{1}{2}$ hours to seconds
   c. $9{,}500$ millilitres to litres
   d. $7{,}000{,}000$ centimetres to kilometres
   e. $8{,}600$ kilogrammes to tonnes
   f. $3\frac{1}{2}$ tonnes to grammes

Solutions:

a. To convert 5 km to m, use the conversion factor $1{,}000\text{m} = 1\text{km}$. Multiply to convert a larger unit (km) to a smaller unit (m):
$5\text{ km} = 5 \times 1{,}000\text{m} = 5{,}000\text{m}$

b. To convert $2\frac{1}{2}$ hours to seconds, use the conversion factor 1 hour = 60 minutes and the conversion factor 1 minute = 60 seconds. These can be combined to give 1 hour = 60×60 seconds = 3600 seconds.

$$\begin{aligned} 2\tfrac{1}{2}\text{hr} &= 2\tfrac{1}{2} \times 3{,}600 \text{ seconds} \\ &= \tfrac{5}{2} \times 3{,}600 \text{ seconds} \\ &= 9{,}000 \text{ seconds} \end{aligned}$$

c. To convert $9{,}500$ millilitres to litres, use the conversion factor $1{,}000\text{ ml} = 1\text{ l}$. Divide to convert a smaller unit (ml) to a larger unit (l).

$$\begin{aligned} 9{,}500\text{ml} &= \tfrac{9{,}500}{1{,}000}\text{l} \\ &= 9.5\text{l} \end{aligned}$$

d. To convert $7{,}000{,}000$ centimetres to kilometres, use multiple conversion factors. These can be combined (as in problem b.) or applied in 2 steps, shown below.
Step 1. Change centimetres to metres using the conversion factor $100\text{ cm} = 1\text{ m}$

Divide: $7{,}000{,}000 \text{ cm} = \frac{7{,}000{,}000}{100} \text{ m} = 70{,}000 \text{ m}$

Step 2. Change metres to kilometres using the conversion factor $1{,}000 \text{ m} = 1 \text{ km}$

Divide: $70{,}000 \text{ m} = \frac{70{,}000}{1{,}000} \text{ km} = 70\text{km}$

Answer: $7{,}000{,}000 \text{ cm} = 70 \text{ km}$

e. To convert $8{,}600$ kilogrammes to tonnes, use the conversion factor $1{,}000 \text{ kg} = 1 \text{ t}$. Divide to convert a smaller unit (kg) to a larger unit (tonnes)

$$\begin{aligned} 8{,}600\text{kg} &= \frac{8{,}600}{1{,}000}\text{t} \\ &= 8.6\text{t} \end{aligned}$$

f. To convert $3\frac{1}{2}$ tonnes to grammes, use 2 steps. Apply multiplication to convert a larger unit (tonnes) to a smaller unit (g).

Step 1. Change tonnes to kilogrammes using the conversion factor $1{,}000 \text{ kg} = 1 \text{ t}$.

$$\begin{aligned} 3\frac{1}{2}t &= 3\frac{1}{2} \times 1{,}000 \text{ kg} \\ &= \frac{7}{2} \times 1{,}000 \text{ kg} \\ &= 3{,}500 \text{ kg} \end{aligned}$$

Step 2. Change kg to g using the conversion factor $1{,}000 \text{ g} = 1 \text{ kg}$.

$$\begin{aligned} 3{,}500 \text{ kg} &= 3500 \times 1{,}000 \text{ g} \\ &= 3{,}500{,}000 \text{ g} \end{aligned}$$

Answer: $3\frac{1}{2} \text{tonnes} = 3{,}500{,}000 \text{ grammes}$

2. The speed of a car is 120km/h. What is the speed in m/sec?

Solution:

To convert a rate, convert the units in both the numerator and denominator. In this problem, we want to convert kilometres to metres and hours to seconds.

120 km/h	$=$	$120\frac{km}{h}$	
	$=$	$120\frac{km}{h} \times 1{,}000$	Convert km to metres
	$=$	$120{,}000\frac{m}{h}$	
	$=$	$\frac{120{,}000\frac{m}{h}}{3{,}600}$	Convert hours to seconds
	$=$	$33\frac{1}{3} m/sec$	Simplify

3. The population density of a town is $4{,}000 \text{ people/km}^2$. Find the number of people if the area of the town is $5{,}000 \text{ m}^2$.

Solution:

The unit for Population density is $\frac{\text{people}}{\text{km}^2}$ and the unit for Area is m^2, so convert the area to km^2. The conversion factor must be applied twice, because the unit is squared.

Step 1. Convert m^2 to km^2:

$$\begin{aligned} 5{,}000 \text{ m}^2 &= 5{,}000\left(\frac{1}{1{,}000} \times \frac{1}{1{,}000}\right)\text{km}^2 \\ &= \frac{1}{200} \text{ km}^2 \end{aligned}$$

Step 2. Multiply to find the number of people in the town:

$$\begin{aligned} \text{Number of people} &= 4{,}000 \frac{\text{people}}{\text{km}^2} \times \frac{1}{200}\text{km}^2 && \text{Population density} \times \text{Area} \\ &= 20 \text{ people} \end{aligned}$$

4. A tank contains $72{,}000 \text{ cm}^3$ of fuel. If one litre cost $\text{Le } 6{,}000.00$, what is the total cost of fuel in the tank?

Solution:

The cost of fuel is $\text{Le } 6{,}000/\text{litre}$ and the quantity of fuel is in cm^3, so convert cm^3 to litres. Use the fact that there are 1,000 cm^3 in each litre.

Step 1. Calculate the litres in the tank:

$$\begin{aligned} 72{,}000 \text{ cm}^3 &= \frac{72{,}000}{1{,}000}\text{l} && 1{,}000 \text{ cm}^3 = 1 \text{ litre} \\ &= 72 \text{ l} && \text{Simplify} \end{aligned}$$

Step 2. Calculate the cost of 72 litres:

$$\begin{aligned} \text{Cost} &= 72\text{l} \times \text{Le } 6{,}000/\text{l} \\ &= \text{Le } 432{,}000.00 \end{aligned}$$

Practice

1. Convert the following as indicated:
   a. 5.3 kilometres to metres
   b. 2 hours 15 minutes to seconds
   c. $9{,}500$ millilitres to litres
   d. $4\frac{1}{2}$ litres to cubic centimetres
   e. $65{,}000{,}000$ centimetres to kilometres
   f. $76{,}500$ kilogrammes to tonnes
   g. $2\frac{3}{4}$ tonnes to grammes
2. The speed of a car is 120 m/sec, what is the speed in km/hr?

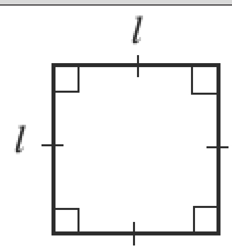
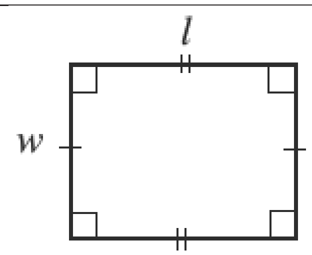
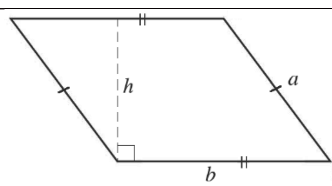
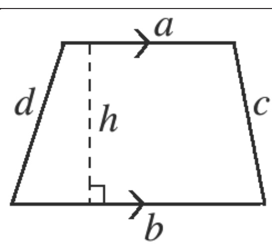
3. A tank contains $16\ \mathrm{m}^3$ of fuel. If one litre cost $\mathrm{Le}\ 6{,}000.00$, what is the total cost of fuel in the tank?
4. The speed of a wave is $3 \times 10^8\ \mathrm{m}/sec$. Find the distance it travels in metres in 2×10^{-6} hours, leaving your answer in standard form.

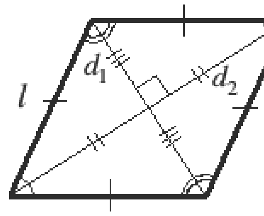
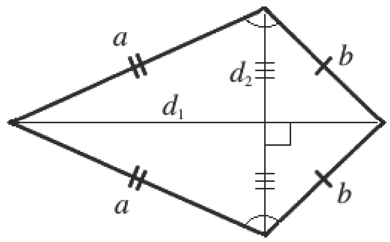
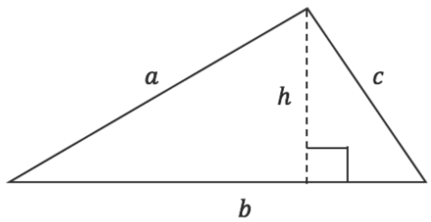
Lesson Title: Area and perimeter of triangles and quadrilaterals	**Theme:** Mensuration
Practice Activity: PHM4-L056	**Class:** SSS 4

Learning Outcome
By the end of the lesson, you will be able to calculate the area and perimeter of triangles and quadrilaterals.

Overview

This lesson is on perimeter and area of several types of quadrilaterals and triangles. Perimeter is the distance around the outside of a shape, and area is the size of the space inside the shape. Use the formulae in the table below to solve the problems.

Shape	Diagram	Perimeter	Area
Square		$P = l + l + l + l = 4l$	$A = l \times l = l^2$
Rectangle		$P = l + l + w + w = 2l + 2w$	$A = l \times w$
Parallelogram		$P = a + a + b + b = 2a + 2b$	$A = b \times h$
Trapezium		$P = a + b + c + d$	$A = \frac{1}{2}(a + b)h$

Rhombus		$P = l + l + l + l = 4l$	$A = \frac{1}{2} d_1 \times d_2$
Kite		$P = a + a + b + b = 2a + 2b$	$A = \frac{1}{2} d_1 \times d_2$
Triangle		$P = a + b + c$	$A = \frac{1}{2} b \times h$

Solved Examples

1. The perimeter of a rectangle is twice that of a square with sides 6 cm. Find the area of the rectangle, if its length is 15 cm.

   Solution:

   To find the area of the rectangle, you need to know its length and width. The length of the rectangle is given, the width can be found from the relation given: perimeter of rectangle = twice perimeter of square.

   Set up an equation where l and w are the dimensions of the rectangle, and a is the side of the square.

$$\begin{aligned} \text{Perimeter of rectangle} &= 2 \times \text{perimeter of square} \\ 2l + 2w &= 2 \times 4a \\ 2(15) + 2w &= 2 \times 4 \times 6 && \text{Substitute } l = 15, a = 6 \\ 2w &= 48 - 30 && \text{Solve for } w \\ 2w &= 18 \\ w &= 9 \text{ cm} \end{aligned}$$

   Use the width to find the area:

$$\begin{aligned} \text{Area of rectangle} &= l \times w \\ &= 15 \times 9 \\ &= 135 \text{ cm}^2 \end{aligned}$$

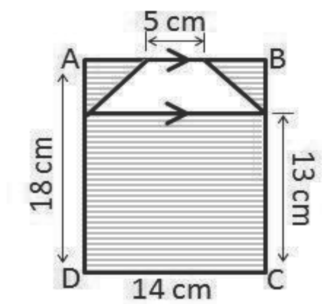
2. A kite with diagonals 4 mm and 10 mm is cut from a piece of rectangular cardboard of dimensions 15 mm$\times$ 25 mm. Find the area of the remaining part of the cardboard.
   Solution:
   To find the area of the remaining part of the cardboard, subtract the area of the kite from the area of the cardboard.

$$\text{Area of cardboard} = 15 \times 25 = 375 \text{ mm}^2$$

$$\text{Area of kite} = \frac{1}{2}(d_1 \times d_2) = \frac{1}{2}(4 \times 10) = 20 \text{ mm}^2$$

$$\text{Area of remaining part} = \text{Area of cardboard} - \text{Area of kite} = 375 \text{ mm}^2 - 20 \text{ mm}^2 = 355 \text{ mm}^2$$

3. Find the area of the shaded portion of rectangle ABCD.
   Solution:
   Note that to find the area of the shaded portion, we need to subtract: Area of Rectangle – Area of Trapezium



Area of Rectangle $= 18 \times 14 = 252 \text{ cm}^2$

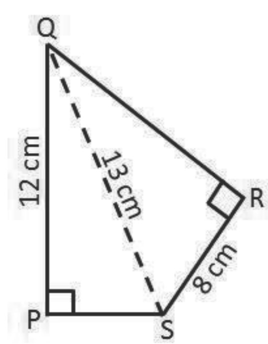
Area of Trapezium $= \frac{1}{2}(a+b)h$

Note that $h = 18 \text{ cm} - 13 \text{ cm} = 5 \text{ cm}$

Area of Trapezium $= \frac{1}{2}(14+5) \times 5 = \frac{1}{2}(19) \times 5 = 47.5 \text{ cm}^2$

$\therefore$ Area of shaded part $= 252 \text{ cm}^2 - 47.5 \text{ cm}^2 = 204.5 \text{ cm}^2$

4. Find the area of the figure PQRS.
   Solution:
   Note that Area of PQRS=Area of ΔPQS+ΔRQS



From the figure, the two triangles are right angled triangles.
First find the sides |PS| and |QR| using Pythagoras' theorem:

From ΔPQS, $QS^2 = PQ^2 + PS^2$

$13^2 = 12^2 + PS^2$

$PS = \sqrt{13^2 - 12^2} = 5 \text{ cm}$

Area of ΔPQS $= \frac{1}{2}|PQ| \times |PS|$

$= \frac{1}{2} \times 12 \times 5$

$= 30 \text{ cm}^2$

From ΔQRS, $QS^2 = QR^2 + RS^2$

$13^2 = 8^2 + QR^2$

$QR = \sqrt{13^2 - 8^2} = 10.25 \text{ cm}$

Area of ΔQRS$= \frac{1}{2}|QR| \times |RS|$

$= \frac{1}{2} \times 10.25 \times 8$

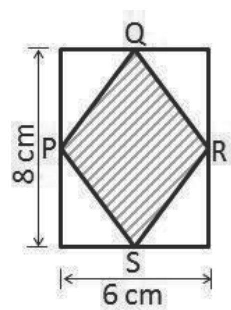
$= 41 \text{ cm}^2$

Area of PQRS =Area of ΔPQS+Area of ΔQRS

$= 30 \text{ cm}^2 + 41 \text{ cm}^2 = 71 \text{ cm}^2$

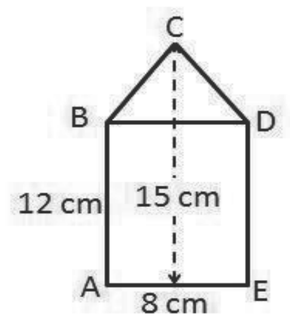
Practice

1. A trapezium with parallel sides 8 cm and 12 cm and distance between them 6 cm is cut from a rectangular piece of cardboard which measures 12 cm by 18 cm. Find the area of the remaining part.



2. In the diagram at right, PQRS are the mid-points of a rectangular surface which measures 6 cm by 8 cm. Find the area of the shaded part.

3. The perimeter of a rectangle is three times that of a square with sides 4 cm. If the length of the rectangle is 14 cm, find its area.



4. Find the area of the figure ABCDE, shown at right:

Lesson Title: Trigonometric ratios	**Theme:** Trigonometry
Practice Activity: PHM4-L057	**Class:** SSS 4

Learning Outcome

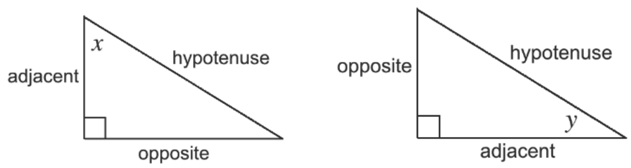
By the end of the lesson, you will be able to identify trigonometric and inverse trigonometric ratios and use them to solve for sides and angles of a triangle.

Overview

This lesson is on trigonometric and inverse trigonometric functions. Trigonometric functions are used to find missing sides in a right-angled triangle, and inverse trigonometric functions are used to find missing angles.

We use 3 types of sides (adjacent, opposite and hypotenuse) in trigonometric ratios. "Adjacent" and "opposite" are determined by their relationship to the angle in question.

For example, consider angles x and y, and the side labels of each triangle:



The 3 trigonometric ratios are:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\text{O}}{\text{H}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\text{A}}{\text{H}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\text{O}}{\text{A}}$$

Sine, cosine, and tangent are functions of angles. The theta symbol (θ) is shown here, and it is often used to represent angles.

We use the term SOHCAHTOA as a way of remembering the ratios:

- SOH stands for "sine equals opposite over hypotenuse".
- CAH stands for "cosine equals adjacent over hypotenuse".
- TOA stands for "tangent equals opposite over adjacent".

Trigonometric tables can be used to find the trigonometric function of a given angle to 4 decimal places. If you do not have access to trigonometric tables, a calculator may be used.

Trigonometric ratios have the relationship $\tan\theta = \frac{\sin\theta}{\cos\theta}$, which may be used to solve problems.

There are some "special angles" which commonly occur and can be used to solve problems. These angles are $30°$, $45°$, $60°$, and $90°$. The trigonometric function of each is shown in the table:

θ	30°	45°	60°	90°
$\sin\theta$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Not defined

The inverse of a function is its opposite. It's another function that can undo the given function. Inverse functions are shown with a power of -1. For example, inverse sine is $\sin^{-1} x$. Inverse sine "undoes" sine: $\sin^{-1}(\sin\theta) = \theta$. The other inverse trigonometric functions are $\cos^{-1} x$ and $\tan^{-1} x$. The inverse functions are also sometimes called "arcsine", "arccosine", and "arctangent".

You can use inverse trigonometric functions to find the degree measure of an angle. You can use the trigonometric tables ("log books") or calculators. You will work backwards using trigonometric tables. Find the decimal number in the chart, and identify the angle that it corresponds to.

At times, you can solve inverse trigonometry problems without a trigonometry table or calculator. This is generally only true for special angles. For example, if you see $x = \sin^{-1}(\frac{1}{2})$, you may identify that $\frac{1}{2}$ is the sine ratio of the special angle 30°. Therefore, the answer would be $x = 30°$.

Solved Examples

1. Find the following using trigonometry tables ("log books"):
    a. $\sin^{-1}(0.5150)$
    b. $\cos^{-1}(0.7891)$

   Solutions:

   a. Solution using a **sine table:**
   - Find 0.5150 in the trigonometric table for sine.
   - It is in row 31, under the first column (.0). This means that the angle has measure 31.0°.

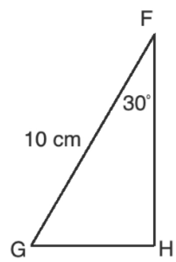
   b. Solution using a **cosine table:**
   - Look for 0.7891 in the cosine table. It is in row 37, under the column for .9.
   - This gives us the angle 37.9°.
2. Find x if $x = \sin^{-1}(\frac{\sqrt{3}}{2})$.

Solution:
Since the inverse sine is the opposite of sine, we can take the sine of both sides to eliminate it.

$$\begin{aligned}\sin x &= \sin(\sin^{-1}(\tfrac{\sqrt{3}}{2}))\\ \sin x &= \frac{\sqrt{3}}{2}\end{aligned}$$

We know from the lesson on special angles that $\sin 60° = \frac{\sqrt{3}}{2}$. Therefore, the answer is $x = 60°$.

3. Find the measures of FH and GH:



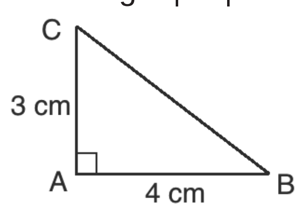
Solution:
Find $|FH|$:

$$\begin{aligned}\cos 30° = \frac{|FH|}{10} &= \frac{\sqrt{3}}{2}\\ 2|FH| &= 10\sqrt{3}\\ |FH| &= \frac{10\sqrt{3}}{2}\\ |FH| &= 5\sqrt{3}\end{aligned}$$

Find $|GH|$:

$$\begin{aligned}\sin 30° = \frac{|GH|}{10} &= \frac{1}{2}\\ |GH| &= \frac{10}{2}\\ |GH| &= 5\end{aligned}$$

4. Find the measure of angle B and the length $|BC|$:



Solution:
Step 1. Identify which function to use. The opposite and adjacent sides are known, so we will use $\tan^{-1}$.

Step 2. Find the tangent ratio. This is the ratio that you will "undo" with $\tan^{-1}$ to find the angle:

$$\tan B = \frac{3}{4} = 0.75$$

Step 3. Find $\tan^{-1}$ of both sides to find the angle measure:

$$\begin{aligned} \tan B &= 0.75 \\ \tan^{-1}(\tan B) &= \tan^{-1}(0.75) \\ B &= \tan^{-1}(0.75) \end{aligned}$$

Calculate $\tan^{-1}(0.75)$ using either a calculator or trigonometry table.

To calculate $\tan^{-1}(0.75)$ using the tangent table: Look for 0.75 in the table. It is not there, but 0.7481 is there. If we add 0.0018 to 0.7481, it will give us 0.75. Find 18 in the "add differences" table, and it corresponds to 7. Therefore, the angle is 36.87.

Answer: $B = 36.87°$

Use Pythagoras' theorem to find $|BC|$

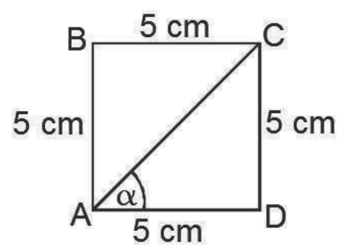
$BC^2 = AC^2 + AB^2 = 3^2 + 4^2 = 25$

$|BC| = \sqrt{25} = 5$ cm

5. The side of a square is 5 cm. Find, correct to two decimal places, the length of a diagonal and the angle between the diagonal and a side of the square.

   Solution:

   Step 1. Draw a diagram. →

   Step 2. Use Pythagoras' theorem to find the diagonal:

   $d^2 = 5^2 + 5^2 = 50$

   $d = \sqrt{50} = 7.07$ cm

   Step 3. Use trigonometry to find the angle:

   If the angle between the diagonal and a side is α, then:

   $\tan \alpha = \frac{5}{5} = 1$

   $\alpha = \tan^{-1} 1 = 45°$

   Answer: The length of a diagonal is 7.07 cm, and the angle between a diagonal and a side is 45°.

Practice

1. Find the following using trigonometry tables:
   a. $\sin^{-1}(0.7250)$; b. $\cos^{-1}(0.6397)$; c. $\tan^{-1}(0.6827)$
2. Find x if: a. $x = \tan^{-1}\frac{1}{\sqrt{3}}$, b. $x = \sin^{-1}\frac{\sqrt{3}}{2}$, c. $x = \cos^{-1}\frac{1}{2}$, d. $x = \sin^{-1}(0.3423)$
3. In right-angle triangle ABC, $\angle C = 90°, \angle A = 70°$ and |AB| = 5 cm. Find the other sides and angle of the triangle.

4. The two legs of a right-angle triangle are 8 cm and 6 cm. Find the hypotenuse side and the angles of the triangle.
5. The dimensions of a rectangle are 7 cm and 5 cm. Find: a. The length of a diagonal; b. The angle between the diagonal and the 7 cm side.

Lesson Title: Solving right-angled triangles	**Theme:** Trigonometry
Practice Activity: PHM4-L058	**Class:** SSS 4

Learning Outcome By the end of the lesson, you will be able to apply the Pythagorean theorem and trigonometric ratios to solve for sides and angles of right-angled triangles, including word problems.

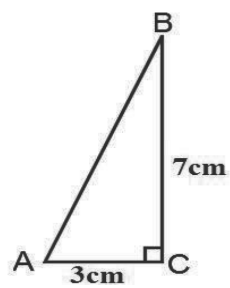
Overview

To "solve" a triangle means to find any missing side or angle measures. You are familiar with several methods for solving triangles, including: trigonometric and inverse trigonometric functions, Pythagoras' theorem, and finding angle measures by subtracting from 180°.

When you have a triangle with missing sides and angles, you need to decide how to solve for them. In some cases, you could solve a problem using different methods. For example, in some cases the side of a right-angled triangle could be solved with Pythagoras' theorem or trigonometry. Choose the method you prefer, or the one that is best for the given problem.

Solved Examples

1. Solve the triangle ABC in which ∠C= $90°$, $a = 7$ cm and $b = 3$ cm.



Solution:

Note that to solve a triangle means finding all the unknown side(s) and angle(s).

Use Pythagoras' theorem to find the side c, since the other two sides are known;

c^2	$=$	$a^2 + b^2$	Pythagoras' theorem
c^2	$=$	$7^2 + 3^2$	Substitute values for a and b
c^2	$=$	58	
c	$=$	$\sqrt{58}$	
c	$=$	7.62 cm	

To find any of the missing angles, use any of the trigonometric ratios to find 1 angle. Then, subtract the known angles from $180°$ to find the other.

$\tan A$	$=$	$\frac{7}{3}$	Apply the tangent ratio to A
$\tan A$	$=$	2.3333	Substitute values for a and b

$$A = \tan^{-1} 2.3333 \quad \text{Take the inverse tangent of both sides}$$
$$A = 66.8°$$

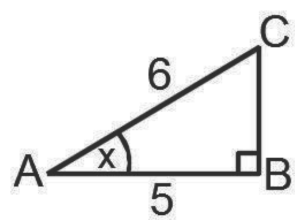
2. If $\cos x = \frac{5}{6}$, where $0° < x < 90°$, find:

   a. $\sin x$ b. $\tan x$ c. $\frac{1+\sin^2 x}{1-\tan^2 x}$

   Solution:

   Given one trigonometric ratio, draw a right-angled triangle. Find the remaining side of the triangle and use it to find the other trigonometric ratios.

   Step 1. Draw a diagram. →



   Step 2. Find the unknown side:

$$|BC|^2 = 6^2 - 5^2$$
$$|BC|^2 = 36 - 25$$
$$|BC| = \sqrt{11}$$

   Step 3. Substitute and evaluate each trigonometric ratio in the problem.

   a. $\sin x = \frac{opp}{hyp} = \frac{\sqrt{11}}{6}$

   b. $\tan x = \frac{opp}{adj} = \frac{\sqrt{11}}{5}$

   c. Substitute and evaluate:

$$\frac{1+\sin^2 x}{1-\tan^2 x} = \frac{1+\left(\frac{\sqrt{11}}{6}\right)^2}{1-\left(\frac{\sqrt{11}}{5}\right)^2} \quad \text{Substitute the values for } \sin x \text{ and } \tan x$$
$$= \frac{1+\frac{11}{36}}{1-\frac{11}{25}} \quad \text{Simplify}$$
$$= \frac{\frac{36+11}{36}}{\frac{25-11}{25}}$$
$$= \frac{47}{36} \times \frac{25}{14}$$
$$= \frac{1175}{504}$$

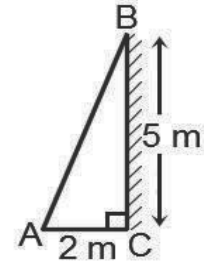
3. A stick leans on a vertical wall of height 5 m. If the foot of the stick is 2 m from the base of the wall, find the:

   a. Length of the stick.

   b. Angle between the stick and the wall.

   Solutions:

   Step 1. Draw a diagram. →



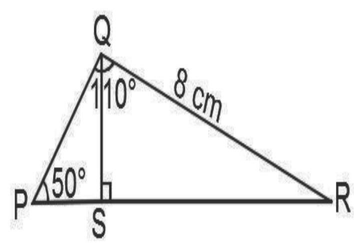
   a. From the diagram, apply Pythagoras' theorem to find the length of the stick, which is AB.

$$|AB|^2 = 2^2 + 5^2 \quad \text{Apply Pythagoras' Theorem}$$
$$= 29 \quad \text{Simplify}$$
$$|AB| = \sqrt{29}$$
$$= 5.39 \text{ m} \quad \text{Length of stick}$$

b. To find the angle between the wall and the stick, find ∠B using any of the trigonometric ratios.

$$\tan B = \frac{2}{5} \quad \text{Apply the tangent ratio}$$
$$= 0.4$$
$$B = \tan^{-1} 0.4 \quad \text{Take the inverse tangent}$$
$$= 21.8° \quad \text{Angle between wall and stick}$$

4. In the figure PQRS, $\angle P = 50°$, $\angle PQR = 110°$, $|QR| = 8$ cm and QS is perpendicular to PR. Find, correct to one decimal place, the length of |PR| of the figure.



Solution:

Note that |PR| = |PS| + |SR|. Find the lengths of PS and SR using trigonometric ratios, then add them to find |PR|.

Step 1. Find ∠SQR, which with the side length $|QR| = 8$ cm, can be used in a trigonometric ratio to find $|SR|$.

$$\angle PQS = 90° - 50° \quad \angle QPS + \angle PQS = 90° \text{ (complementary angles)}$$
$$= 40°$$
$$\angle SQR = 110° - \angle PQS$$
$$= 110° - 40°$$
$$= 70°$$

Step 2. Apply the sine ratio to ΔSQR, to find $|SR|$:

$$\frac{|SR|}{8} = \sin \angle SQR \quad \text{Apply the sine ratio to } \angle SQR$$
$$|SR| = 8 \sin 70°$$
$$= 8 \times 0.9397$$
$$= 7.5 \text{ cm} \quad \text{Length of SR}$$

Step 3. To find |PS|, you need either of the sides PQ or SQ. You can find the side SQ from ΔSQR.

$$\frac{|SQ|}{8} = \cos \angle SQR$$
$$|SQ| = 8 \cos 70°$$
$$= 8 \times 0.3420$$
$$= 2.7 \text{ cm}$$

Since SQ is now known, use ΔPQS to find the length of PS.

$$\frac{|PS|}{|SQ|} = \tan\angle PQS \qquad \text{Apply tangent ratio to } \angle PQS \text{ of } \Delta PQS$$

$$|PS| = |SQ|\tan 40°$$

$$= 2.7 \times 0.8391 \qquad \text{Substitute values for } |SQ| \text{ and } \tan 40°$$

$$= 2.3 \text{ cm} \qquad \text{Length of PS}$$

Step 4. Add |PS| and |SR| to find |PR|.

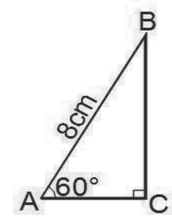
$$|PR| = |PS| + |SR|$$

$$= 2.3 + 7.5$$

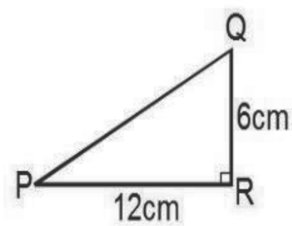
$$= 9.8 \text{ cm} \qquad \text{Length of PR}$$

Practice

1. Find the missing sides and angles in the figure:



2. Find the missing side and angles in the figure:



3. If $\tan x = \frac{2}{5}$, where $\angle x$ is less than $90°$, find:

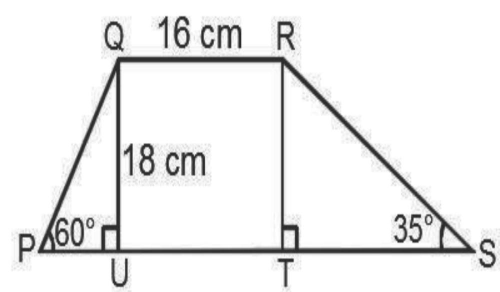
   a. $\sin x$ b. $\cos x$ c. $\sin^2 x + \cos^2 x$ d. $\frac{1+\sin^2 x}{1+\cos^2 x}$

4. A ladder 6 metres long leans against a vertical wall. If the foot of the ladder is 2 metres from the base of the wall, find:

   a. The angle between the ladder and the wall;

   b. The height of the wall.

5. Find the length |PS| in the figure at right:



Lesson Title: Angles of elevation and depression	Theme: Trigonometry
Practice Activity: PHM4-L059	**Class:** SSS 4

Learning Outcome
By the end of the lesson, you will be able to solve practical problems related to angles of elevation and depression.

Overview

"Elevation" is related to height. Problems on angles of elevation handle the angle that is associated with the height of an object. An angle of elevation is measured a certain distance away from an object.

Angle of elevation problems generally deal with 3 measures: the angle, the distance from the object, and the height of the object. You may be asked to solve for any of these measures. These can be solved for using trigonometry to find distances, and inverse trigonometry to find angles.

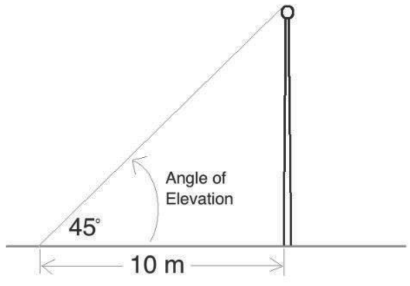
"Depression" is the opposite of elevation. "Depressed" means downward. So, if there is an angle of depression, it is an angle in the downward direction.

The angle of depression is the angle made with the **horizontal** line. See the diagram in Solved Example 2. The horizontal line is at the height of the cliff.

Angle of depression problems generally deal with 3 measures: the angle, the horizontal distance, and the depth of the object. Depth is the opposite of height. It is the distance downward. You may be asked to solve for any of these 3 measures.

Solved Examples

1. At a point 10 metres away from a flag pole, the angle of elevation of the top of the pole is 45°. What is the height of the pole?
   Solution:
   First, draw a diagram:



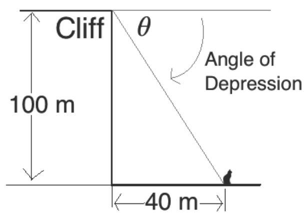
Solve using the tangent ratio, because we are concerned with the sides opposite and adjacent to the angle.

$$\begin{aligned} \tan 45° &= \frac{h}{10} && \text{Set up the equation} \\ 1 &= \frac{h}{10} && \text{Substitute } \tan 45° = 1 \\ 10\text{ m} &= h \end{aligned}$$

2. A cliff is 100 metres tall. At a distance of 40 metres from the base of the cliff, there is a cat sitting on the ground. What is the angle of depression of the cat from the cliff?

Solution:

First, draw a diagram:



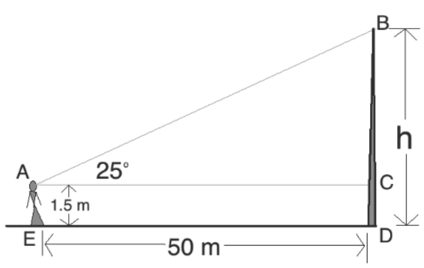
Solve using the tangent ratio, because we are concerned with the sides opposite and adjacent to the angle.

$$\begin{aligned} \tan\theta &= \frac{100}{40} = 2.5 && \text{Set up the equation} \\ \tan^{-1}(\tan\theta) &= \tan^{-1}(2.5) && \text{Take the inverse tangent} \\ \theta &= 68.2 && \text{Use the tangent tables} \end{aligned}$$

3. A woman standing 50 metres from a flag pole observes that the angle of elevation of the top of the pole is 25°. Assuming her eye is 1.5 metres above the ground, calculate the height of the pole to the nearest metre.

Solution:

First, draw a diagram:



To find the height of the flag pole, we must find the length of BC, then add it to the height of the woman's eye (CD or AE), which is 1.5 metres.

Step 1. Find $\overline{BC}$:

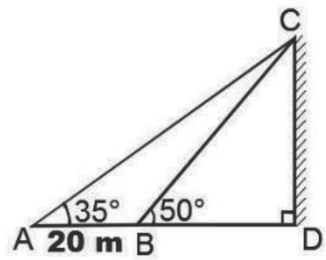
$$\begin{aligned} \tan 25° &= \frac{\overline{BC}}{50} && \text{Set up the equation} \\ 0.4663 &= \frac{\overline{BC}}{50} && \text{Substitute } \tan 25° = 0.4663 \text{ (from table)} \end{aligned}$$

$$50 \times 0.4663 = \overline{BC} \quad \text{Multiply throughout by 50}$$

$$\overline{BC} = 23.315 \quad \text{metres}$$

Step 2. Add: $h = BC + CD = 23.315 + 1.5 = 24.815$
Rounded to the nearest metre, the height of the pole is 25 m.

4. Two points A and B, 20 metres apart, are on the same horizontal line as the foot of a building. The angles of elevation of the top of the building from A and B are $35°$ and $50°$, respectively. If point B is closer to the building than A, find:
   a. The distance between point B and the building.
   b. The height of the building.



Solution:
From the diagram, the height of the building is |CD|, let that be h. The distance of point B from the building is |BD|, let that be x. The distance of A from the building is $20 + x$. We can set up 2 equations with h and x, and solve them to find the answers.

Equation (1):

$$\frac{h}{x} = \tan 50° \quad \text{Applying the tangent ratio to } \angle B \text{ in } \Delta BCD$$

$$h = x \tan 50°$$

Equation (2):

$$\frac{h}{20+x} = \tan 35° \quad \text{Applying the tangent ratio for } \angle A \text{ in } \Delta ACD$$

$$h = (20 + x)\tan 35°$$

a. Set the 2 equations equal, and solve for x:

$$\begin{aligned}
x \tan 50° &= (20 + x)\tan 35° \\
x \tan 50° &= 20 \tan 35° + x \tan 35° \\
x \tan 50° - x \tan 35° &= 20 \tan 35 \\
x(\tan 50° - \tan 35°) &= 20 \tan 35 \\
x &= \frac{20 \tan 35}{\tan 50 - \tan 35} \\
&= 28.47 \text{ m}
\end{aligned}$$

The distance between point B and the building is 28.47 metres.

b. Use equation 1 to find the height of the building

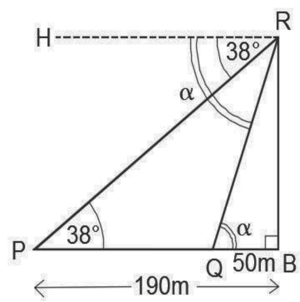
$$\begin{aligned}
h &= x \tan 50° \\
&= (28.47)\tan 50° \\
&= (28.47)(1.192) \\
&= 33.94 \text{ m}
\end{aligned}$$

The height of the building is 33.94 metres.

5. From the top of a cliff, the angle of depression of a boat P on sea, which is 190 m from the foot of the cliff B, is $38°$. Q is another boat on the same horizontal line PB such that |QB| is 50 m. Find, correct to one decimal place:
   a. The height of the cliff.
   b. The angle of depression of the boat Q from the top of the cliff.

   Solutions:

   Draw a diagram (see below). From the diagram, the height of the cliff is |RB|, let that be h. Let ∠RQB, which is the same as $\angle HRQ$, the angle of depression, be α.



   a. To find the height of the cliff, use the angle at P and the distance of P from the cliff.

$$\frac{h}{190} = \tan 38°$$
$$h = 190 \tan 38°$$
$$= 190 \times 0.7813$$
$$= 148.4 \text{ m} \qquad \text{Height of the cliff}$$

   b. To find α, use the height of the cliff and the distance of Q from the base of the cliff.

$$\tan\alpha = \frac{h}{50}$$
$$\tan\alpha = \frac{148.4}{50}$$
$$\tan\alpha = 2.968$$
$$\angle\alpha = \tan^{-1} 2.968$$
$$= 71.38° \qquad \text{Angle of depression of boat Q from the top of the cliff.}$$

Practice

1. There is a dog sitting 60 metres from the base of a cliff. If the angle of depression of the dog from the top of the cliff is 45°, how tall is the cliff?
2. Point A is on the ground 10 metres away from a tree. If the tree is 12 metres tall, what is the angle of elevation at point A?
3. A man 1.6 m tall is standing 15 m from a tree that is 25 m tall. Find, correct to 1 decimal place, the angle of elevation of the top of the tree observed by the man.
4. Two points M and N, 32m apart, are on the same horizontal line as the foot of a building. The angles of elevation of the top of the building from M and N are $25°$ and $48°$ respectively. If point N is closer to the building than M, find correct to 3 significant figures:
   a. The distance between point N and the foot of the building.
   b. The height of the building.

5. Boat A is on the sea, 150 m from the foot of the cliff, F. The angle of depression from the top of the cliff to boat A is $41°$. Boat B is on the same horizontal line AF such that |BF| is 40 m. Find correct to one decimal place:
    a. The height of the cliff.
    b. The angle of depression of boat B from the top of the cliff.

Lesson Title: The unit circle and trigonometric functions of larger angles	**Theme:** Trigonometry
Practice Activity: PHM2-L060	**Class:** SSS 4

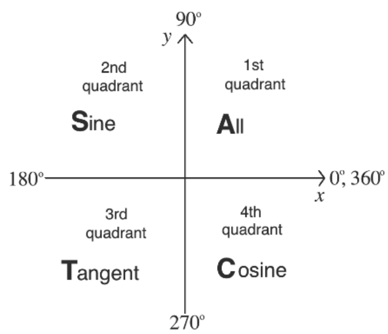
Learning Outcomes

By the end of the lesson, you will be able to:

1. Define $\sin\theta$ and $\cos\theta$ as ratios within a unit circle.
2. Solve problems involving trigonometric functions of obtuse and reflex angles.

Overview

This lesson is on finding the trigonometric ratios of obtuse and reflex angles. These can be found using the trigonometric ratios of corresponding acute angles, and using the chart below:

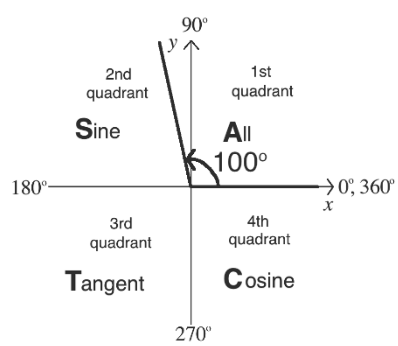


Angles are centered at the origin of this chart, which is the point where the x- and y-axes cross. Positive angles open in the counterclockwise direction. An angle in the first quadrant is acute, and an angle in the second quadrant is obtuse. An angle in the third or fourth quadrant is a reflex angle. The quadrant that an angle lies in tells you whether the result of the trigonometric ratio will be positive or negative.

In the diagram above, the words "All, Cosine, Tangent, and Sine" tell you which ratios are **positive** in the given quadrants. We use the word "ACTS" to remember which ratios are positive. The word ACTS starts in the **first** quadrant and goes in a **clockwise** direction.

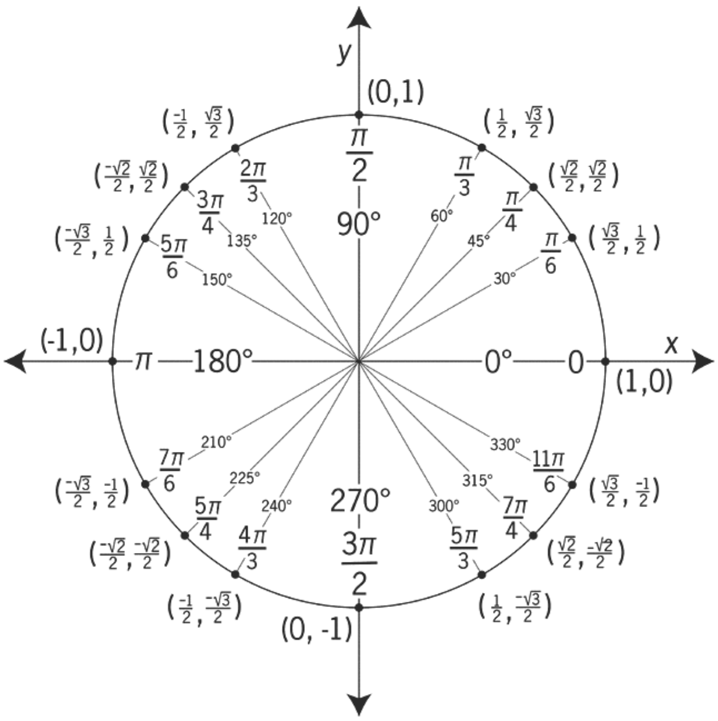
Each obtuse or reflex angle has an **associated acute angle**. This is the acute angle that it forms with the x-axis when it is laid on the 4 quadrants in the diagram above. To find the ratio of an obtuse or reflex angle, find the ratio of the associated acute angle. Then, apply the correct sign for that quadrant.

For example, consider the angle 100° at right.



It forms an 80° angle with the x-axis in the 2nd quadrant. Therefore, to find a trigonometric ratio of 100°, you would find the ratio of 80° and apply the appropriate sign (positive or negative).

The trigonometric ratios of certain angles between 0° and 360° can be found using the unit circle, which is shown below.



Unit Circle[2]

The unit circle is drawn on the Cartesian plane so that the length of its radius is 1 unit. Any point P on the circle forms an angle where each side of the angle is a radius of the circle. Each point P on the circle has coordinates that are an ordered pair. The x-value of the ordered pair is the cosine of the angle formed by P, and the y-value of the ordered pair is the sine of the angle formed by P.

That is, $x = \cos\theta$ and $y = \sin\theta$.

The angles in quadrant 1 of the unit circle are special angles. The angles in other quadrants correspond to the special angles. For example, consider angle 150°. Its

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http://cnx.org/contents/fd53eae1-fa23-47c7-bb1b-972349835c3c@.

corresponding acute angle (the acute angle it forms with the x-axis) is 30°, which is a special angle.

Solved Examples

1. Find $\sin 100°$

   Solution:

   Step 1. Find the sine of the associated acute angle: $\sin 80° = 0.9848$

   Step 2. Keep the positive sign, because sine is positive in the 2nd quadrant:

   $\sin 100° = 0.9848$

2. Find $\cos 165°$

   Solution:

   Step 1. Find the cosine of the associated acute angle: $\cos 15° = 0.9659$

   Step 2. Change the sign to negative, because cosine is negative in the 2nd quadrant:

   $\cos 165° = -0.9659$

3. Find $\tan 230°$

   Solution:

   Step 1. Find the tangent of the associated acute angle: $\tan 50° = 1.192$

   Step 2. Keep the positive sign, because tangent is positive in the 3rd quadrant:

   $\tan 50° = 1.192$

4. Find $\sin 60°$

   Solution:

   Identify 60° on the unit circle. Identify the y-coordinate of the point, $\frac{\sqrt{3}}{2}$.

   $\sin 60° = \frac{\sqrt{3}}{2}$

5. Find $\cos 240°$

   Solution:

   Identify 240° on the unit circle. Identify the x-coordinate of the point, $\frac{-1}{2}$.

   $\cos 240° = \frac{-1}{2}$.

6. If $\sin\theta = -0.5632$, and $0° \leq \theta \leq 360°$, find the possible values of θ.

   Solution:

   Sine is negative in the 3^{rd} and 4^{th} quadrants. Using the corresponding acute angle x, the possible values of θ are i) $(180° + x°)$; ii) $(360° - x°)$.

   From the sine table, $\sin x = 0.5632$, gives $x = \sin^{-1}(0.5632) = 34.28°$.

Therefore, the possible values are:

$$\theta = (180° + 34.28°) = 214.28°$$
$$\theta = (360° - 34.28°) = 325.72°$$

7. Without using calculators or tables, find the value of $\sin 330° + 2\cos 300°$.

Solution:

$330°$ is in the 4^{th} quadrant where sine is negative and $300°$ is also in the 4^{th} quadrant where cosine is positive.

$\sin 330° + 2\cos 300°$	$=$	$-\sin 30° + 2\cos 60$	Using the acute angles in the 4^{th} quadrant
	$=$	$-\frac{1}{2} + 2 \times \frac{1}{2}$	Using special angles $30°$ and $60°$
	$=$	$\frac{1}{2}$	Simplify

8. Find the value of $\sin 120° + \tan 330°$ without using tables or calculators, leaving your answer in surd form.

Solution:

$120°$ is in the 2^{nd} quadrant, where sine is positive and $330°$ is in the 4^{th} quadrant, where tangent is negative.

$\sin 120° + \tan 330°$	$=$	$\sin 60° - \tan 30°$	Using acute angles
	$=$	$\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{3}}$	Using special angles
	$=$	$\frac{\sqrt{3}}{6}$	Simplify

9. Without using tables or calculators, find the value of $\cos^2 300° - 3\tan 330°$, leaving your answer in surd form.

Solution:

$300°$ is in the 4^{th} quadrant, where cosine is positive and $330°$ is also in the 4^{th} quadrant, where tangent is negative.

$\cos^2 300° - 3\tan 330°$	$=$	$(\cos 300°)^2 - 3\tan 330°$	
	$=$	$(\cos 60°)^2 - 3(-\tan 30°)$	Using the acute angles
	$=$	$\left(\frac{1}{2}\right)^2 + 3 \times \frac{1}{\sqrt{3}}$	Using special angles $30°$ and $60°$
	$=$	$\frac{1}{4} + \frac{3}{\sqrt{3}}$	Simplify

Practice

1. Find the value of the following:
    a. $\cos 146°$ b. $\tan 128°$ c. $\sin 330°$ d. $\tan 250°$
2. Find the value of θ, if
    a. $\sin\theta = -0.4$, where $0° \leq \theta \leq 360°$
    b. $5\tan\theta = -12$, where $0° \leq \theta \leq 360°$
    c. $1 + 2\sin\theta = 0$, where $0° \leq \theta \leq 360°$
3. Without using tables or calculators, find the value of the following leaving your answers in surd form:
    a. $\sin 300° - \cos 150°$
    b. $2\tan 150° + \sin 120°$
    c. $\cos^2 300° - 3\tan 330°$

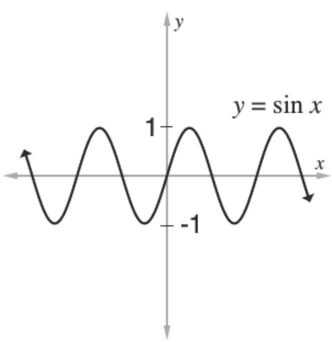
Lesson Title: Graphs of trigonometric functions	**Theme:** Trigonometry
Practice Activity: PHM2-L061	**Class:** SSS 4

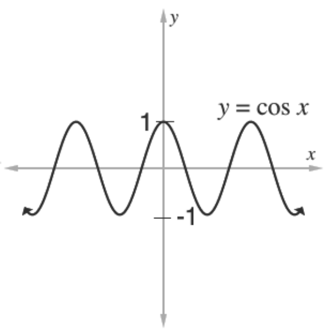
Learning Outcome
By the end of the lesson, you will be able to draw the graph of $\sin\theta$, $\cos\theta$, and functions of the form $y = a\sin\theta + b\cos\theta$.

Overview

This lesson is on graphing the sine and cosine functions. The process is similar to graphing other types of functions. Use a table of values, with degrees as the x-values, and the results of the sine function as the y-values. To graph trigonometric functions, we usually use intervals such as 30° or 45° between x-values. Often, we are asked to graph a trigonometric function over the interval $0° \leq x \leq 360°$ or $0° \leq x \leq 180°$.

You will notice that the sine and cosine curve have the same shape. They both go on forever in both x-directions, and remain between $y = -1$ and $y = 1$. However, they have different starting points. $y = \sin x$ intersects the origin. $y = \cos x$ intersects the y-axis at $y = 1$.





You will also be asked to graph functions that contain both sine and cosine, such as of $y = 2\sin x + \cos x$. The process is similar to graphing the sine or cosine function alone (see Solved Example 3). It is useful to have extra rows in the table to perform calculations. For example, for the function above, you would have a row for calculating $2\sin x$, and a row for calculating $\cos x$. In the last row, you would add them together. The last row are y-values that are used for plotting points. After graphing, the shape of the curve will be similar to what we have seen before with $y = \sin x$ and $y = \cos x$.

Solved Examples

1. Draw the graph of $y = \sin x$ for values of x from 0° to 360°, using intervals of 45°.
   a. Use the graph to solve $\sin x = 0$.

b. Find the truth set of the equation $\sin x = \frac{1}{2}$.

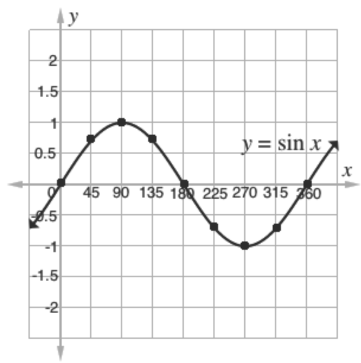
Solution:

First, make a table of values. The x-values in our table of values will be degrees between 0° and 360°. We want intervals of 45°, so add 45° to each x-value to get the next value for the table. Find the sine of each value in the table using the unit circle.

x	$0°$	$45°$	$90°$	$135°$	$180°$	$225°$	$270°$	$315°$	$360°$
$\sin x$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{2}}{2}$	0	$-\frac{\sqrt{2}}{2}$	-1	$-\frac{\sqrt{2}}{2}$	0

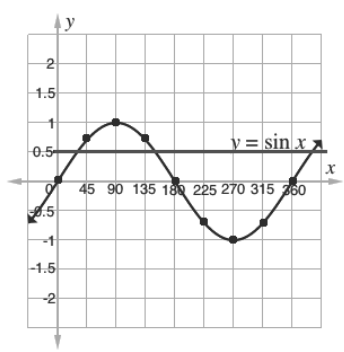
Each column is an ordered pair, which can be plotted on the Cartesian plane. For example: $(0°, 0), (45°, \frac{\sqrt{2}}{2}), (90°, 1)$

Plot each of the points and connect them in a curve:



a. We want to find the places on the graph of sine where $y = 0$ on the interval of concern, $0° \leq x \leq 360°$. This is similar to solving a quadratic equation. We have graphed the function, and we want to find where it crosses the x-axis.
Answers: $x = 0°, 180°, 360°$

b. To find the truth set, we find all points in the given interval where this equation is true. This equation tells us that $y = \frac{1}{2}$. Draw a horizontal line at $y = \frac{1}{2}$, and find all the points at which the line intersects the curve of $y = \sin x$.



Identify the approximate x-values (within $0° \leq x \leq 360°$) at which the line and curve intersect.
Answers: 30°, 150°

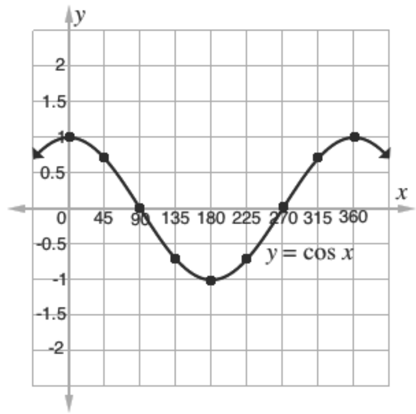
2. Draw the graph of $y = \cos x$ for values of x from 0° to 360°, using intervals of 45°.
   a. Use the graph to solve $\cos x = 0$.
   b. Find the truth set of the equation $\cos x = 1$.

Solution:

First, make a table of values. The x-values in our table of values will be degrees between 0° and 360°. We want intervals of 45°, so add 45° to each x-value to get the next value for the table. Find the cosine of each value in the table using the unit circle.

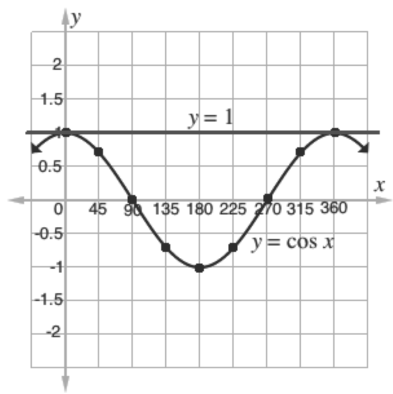
x	$0°$	$45°$	$90°$	$135°$	$180°$	$225°$	$270°$	$315°$	$360°$
$\cos x$	1	$\frac{\sqrt{2}}{2}$	0	$-\frac{\sqrt{2}}{2}$	-1	$-\frac{\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1

Plot each of the points and connect them in a curve:



c. We want to find the places on the graph of cosine where $y = 0$ on the interval of concern, $0° \leq x \leq 360°$.
Answers: $x = 90°, 270°$

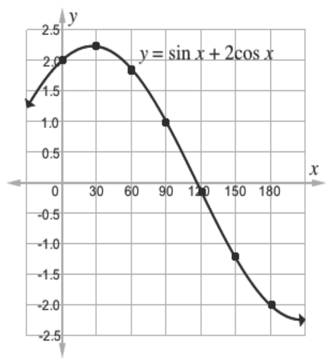
b. To find the truth set, we find all points in the given interval where this equation is true. This equation tells us that $y = 1$. Draw a horizontal like at $y = 1$, and find all the points at which the line intersects the curve of $y = \cos x$.
Answers: $x = 0°, 360°$



3. Draw the graph of $y = 2\sin x + \cos x$ for values of x from 0° to 180°, using intervals of 30°. Use the graph to approximate the solution of $\sin x + 2\cos x = 0$.
Solution:
First, make a table of values. The x-values in our table of values will be degrees between 0° and 360°. We want intervals of 30°, so add 30° to each x-value to get the next value for the table.
Find the sine and cosine of each x-value using the trigonometric tables. Remember to multiply each cosine value by 2. Add the sine and cosine rows to find the values for the last row.

x	$0°$	$30°$	$60°$	$90°$	$120°$	$150°$	$180°$
$\sin x$	0	0.5	0.9	1	0.9	0.5	0
$2\cos x$	2.0	1.7	1.0	0	-1.0	-1.7	-2.0
$\sin x + 2\cos x$	2.0	2.2	1.9	1.0	-0.1	-1.2	-2.0

Plot each of the points and connect them in a curve:



To solve $\sin x + 2\cos x = 0$, we need to find the places on the graph where $y = 0$ on the interval of concern, $0° \leq x \leq 180°$. We can observe that the curve intersects the x-axis just before 120°. A good estimate for the solution is 115°.

Practice

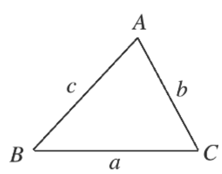
1. Draw the graph of $y = -2\sin x$ for values of x from $0°$ to $180°$. Using intervals of $30°$.
    a. From the graph, find y when $x = 170°$
    b. Find the truth set of the equation $-2\sin x + 1 = 0$.
2. Draw the graph of $y = 2\cos x$ for values of θ from $0°$ to $360°$, using intervals of $45°$. From your graph, find y when $x = 200°$.
3. Draw the graph of $y = \cos^2\theta$ for values of θ from $0°$ to $180°$, using intervals of $30°$. From your graph:
    a. Find y when $\theta = 45°$.
    b. Find the truth set of $\cos^2\theta = 1$.
4. Draw the graph of $y = \sin x + \cos x$ for values of x from $0°$ to $360°$, using intervals of $30°$. Use your graph to find the approximate values of x for which $\sin x + \cos x = 0.75$.
5. Draw the graph of $y = 3\sin x - 2\cos x$ for $0° \leq x \leq 180°$, using intervals of $30°$. From the graph, find:
    a. The truth set of $y = 2$.
    b. The solution of $3\sin x - 2\cos x = 0$.

Lesson Title: Sine and cosine rules	**Theme:** Trigonometry
Practice Activity: PHM2-L062	**Class:** SSS 4

Learning Outcome
By the end of the lesson, you will be able to use the sine and cosine rules to calculate lengths and angles in triangles.

Overview

The sine rule allows us to solve for missing side of any triangle, as long as we have enough information. The sine rule is given by the ratios $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ for a triangle:



For triangle ABC, the angles are usually labeled with capital letters, while the sides are labeled with lower case letters. Using the **sine rule**, we can solve a triangle if we are given **any 2 angles and 1 side** in the problem, or if we are given **2 sides and the angle opposite 1 of them**.

In some cases, we do not have enough information to use the sine rule to solve a problem. The cosine rule allows us to solve triangles where the sine rule cannot be used. We can use the **cosine rule** if **two sides and the angle between them** are given.

The cosine rule says that for the triangle given above, the following are true:

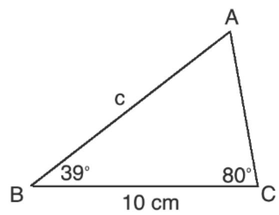
$$c^2 = a^2 + b^2 - 2ab \cos C$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$a^2 = b^2 + c^2 - 2bc \cos A$$

The formulae above are used to solve for sides. The subject of each formula can be changed to solve for the angles:

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$
$$\cos B = \frac{a^2+c^2-b^2}{2ac}$$
$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$

Solved Examples

1. Find the length of missing side c:



Solution:

We can use the sine rule, because we are given 2 angles and 1 side.

Use two fractions from the sine rule: $\frac{a}{\sin A} = \frac{c}{\sin C}$

Substitute known values (a and C) into the formula:

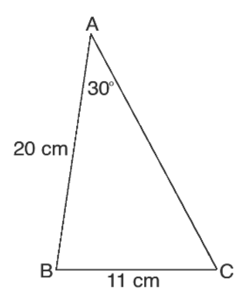
$$\frac{10}{\sin A} = \frac{c}{\sin 80°}$$

There are 2 unknowns. Find A by subtracting the known angles of the triangle from 180: $A = 180° - (39° + 80°) = 61°$

Substitute $A = 61°$ into the formula, and solve for c:

$$\frac{10}{\sin 61°} = \frac{c}{\sin 80°}$$
$$10 \times \sin 80° = c \times \sin 61°$$
$$c = \frac{10 \times \sin 80°}{\sin 61°}$$
$$c = \frac{10 \times 0.9848}{0.8746} \quad \text{Substitute values from the sine table}$$
$$c = 11.26 \text{ cm} \quad \text{Simplify}$$

2. Solve ΔABC, given in the diagram to the right.



Solution:

We can find angle C using the sine rule, because we are given 2 sides and the angle opposite one of them.

We have enough info to find C with the formula: $\frac{a}{\sin A} = \frac{c}{\sin C}$.

Substitute the values and solve:

$$\frac{11}{\sin 30°} = \frac{20}{\sin C} \quad \text{Substitute in the formula}$$
$$11 \times \sin C = 20 \times \sin 30° \quad \text{Cross multiply}$$
$$\sin C = \frac{20 \times \sin 30°}{11} \quad \text{Solve for } C$$
$$\sin C = \frac{20 \times 0.5}{11} = \frac{10}{11}$$
$$\sin C = 0.9091$$

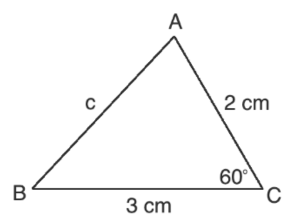
$$
\begin{aligned}
C &= \sin^{-1} 0.9091 && \text{Take the inverse sine of both sides} \\
C &= 65.38° && \text{Use the sine table}
\end{aligned}
$$

Subtract A and C from 180 to find B: $B = 180° - (30° + 65.38°) = 84.82°$

Use the sine formula to find the side |AC| or b:

$$
\begin{aligned}
\frac{b}{\sin 84.82} &= \frac{11}{\sin 30} && \text{Substitute in the formula} \\
b \times \sin 30° &= 11 \times \sin 84.82° && \text{Cross-multiply} \\
b &= \frac{11 \times \sin 84.82°}{\sin 30°} && \text{Solve for } b \\
b &= \frac{11 \times 0.9959}{0.5} = \frac{10.955}{0.5} \\
b &= 21.91 \text{ cm}
\end{aligned}
$$

3. Find the length of missing side c:



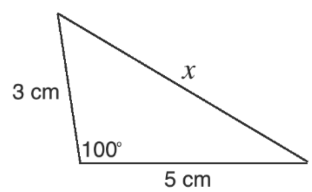
Solution:

There is not enough information to use the sine rule. However, we can use the cosine rule because we have 2 sides and the angle between them.

Apply the formula $c^2 = a^2 + b^2 - 2ab\ \cos C$:

$$
\begin{aligned}
c^2 &= a^2 + b^2 - 2ab\ \cos C && \text{Formula} \\
&= 3^2 + 2^2 - 2(3)(2)\ \cos 60° && \text{Substitute values from triangle} \\
&= 3^2 + 2^2 - 2(3)(2)(0.5) && \text{Substitute } \cos 60° = 0.5 \\
&= 9 + 4 - 12\ (0.5) && \text{Simplify} \\
&= 13 - 6 \\
c^2 &= 7 \\
\mathrm{c} &= \sqrt{7} = 2.65 \text{ cm to 2 d.p.} && \text{Take the square root of both sides}
\end{aligned}
$$

4. Find the length of x in the triangle below:

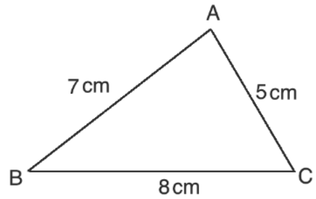


Solution:

We can use the cosine rule because we have 2 sides and the angle between them.

x^2	$=$	$3^2 + 5^2 - 2(3)(5)\cos 100°$	Substitute values from triangle
	$=$	$3^2 + 5^2 - 2(3)(5)(-0.1736)$	Substitute $\cos 100° = -0.1736$
	$=$	$9 + 25 + 5.208$	Simplify
x^2	$=$	39.208	
x	$=$	$\sqrt{39.208}$	Take the square root of both sides
x	$=$	6.26 cm to 2 d.p.	

5. Find the measures of angles A, B, and C using the cosine rule:



Solution:

Use each formula for finding the measure of each angle. Substitute the appropriate values and simplify:

$$\cos A = \frac{5^2+7^2-8^2}{2(5)(7)} = \frac{25+49-64}{70} = \frac{10}{70} = 0.1429$$
$$\cos A = 0.1429$$
$$A = \cos^{-1} 0.1429$$
$$A = 81.8° \text{ to 1 d.p.}$$

$$\cos B = \frac{7^2+8^2-5^2}{2(7)(8)} = \frac{49+64-25}{112} = \frac{88}{112} = 0.7857$$
$$\cos B = 0.7857$$
$$B = \cos^{-1} 0.7857$$
$$B = 38.2° \text{ to 1 d.p.}$$

$$\cos C = \frac{5^2+8^2-7^2}{2(5)(8)} = \frac{25+64-49}{80} = \frac{40}{80} = 0.5$$
$$\cos C = 0.5$$
$$C = \cos^{-1} 0.5$$
$$C = 60°$$

Check your work by adding the angles together:

$A + B + C = 81.8° + 38.2° + 60° = 180°$.

Practice

Find the missing angles or sides in the following triangles. Draw each triangle, then complete the calculations using the appropriate rule or rules.

1. In ΔABC, if $\angle B = 35°, a = 2.7$ m, and $b = 5.1$ m, find $\angle A$ to 3 significant figures.
2. In $\Delta ABC,\ b = 3.8$ m, $\angle A = 25°$, and $a = 1.8$ m. Find $\angle B$ and $\angle C$ to one decimal place.
3. In $\Delta ABC,$ if $a = 15.73$ mm, $\angle B = 121°$, and $c = 23.15$ mm, find b to the nearest millimeter.
4. In $\Delta ABC,$ if $a = 4, b = 7$, and $c = 5,$ find $\angle A$ to the nearest degree.
5. Solve $\Delta ABC,$ given $a = 9.12$ mm, $b = 4.87$ mm and $\angle A = 63°$. Give each result to 3 significant figures.
6. Solve $\Delta PQR,$ given $p = 5, q = 8$ and $r = 10$. Give each result to 1 decimal place.

Lesson Title: Three-figure bearings	**Theme:** Bearings
Practice Activity: PHM4-L063	**Class:** SSS 4

Learning Outcomes
By the end of the lesson, you will be able to:
1. Identify angles measured clockwise from the geographic north.
2. Represent bearings as angles in three digits.
3. Find the reverse bearing of a given bearing.
4. Solve simple problems involving three figure bearings.

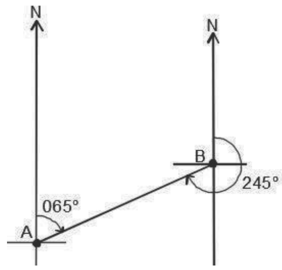
Overview

Three-figure bearings are bearings given in 3 digits. These 3 digits give the angle of the bearing in the clockwise direction from the geographic north. The angles range from 000° to 360°. They must always have 3 digits, even when they're actually less than 100 degrees.

Three-figure bearing diagrams have one arrow pointing in the north direction (shown on the right) and another arrow showing the bearing.

When we talk about "reverse" bearings, we must have 2 points. Consider 2 points A and B. We have the bearing from A to B, and we have the bearing from B to A. These are different, because bearings are about direction. A to B is a different direction than B to A. They are reverse. Reverse bearings are sometimes called "back bearings".

Consider the example:



The bearing from A to B is 065°, and the bearing from B to A is 245°. For both bearings, we use the line that joins them and the north direction. We find the bearing of the line joining them from the north.

Depending on the size of the first bearing, you will add or subtract 180° to find the reverse bearing:

- Reverse bearing = $\theta + 180°$ if θ is less than $180°$
- Reverse bearing = $\theta - 180°$ if θ is more than $180°$

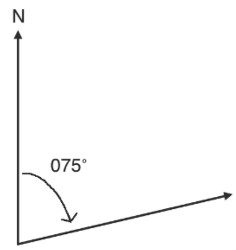
Note that there are 2 ways that bearings can be described in problems. “The bearing of T from S” is the same as “the bearing from S to T”.

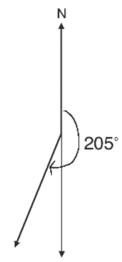
Solved Examples

1. Draw diagrams to show the following bearings:
   a. 009°
   b. 075°
   c. 205°

   Solutions:

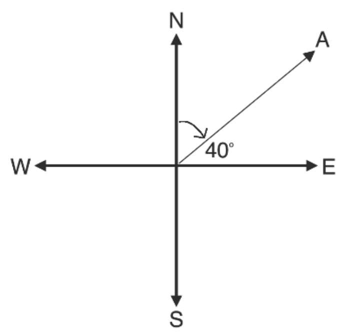
   a.


   b.


   c.


   Note that for bearings greater than 180, the vertical line can be extended down. For part c., the vertical line is extended down and used to measure 25° clockwise from south (because $205° - 180° = 25°$).

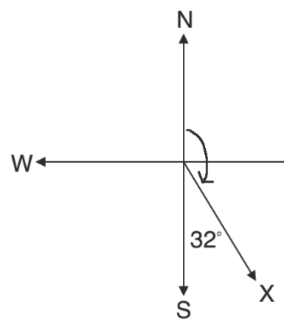
2. Find the three-point bearing of A:



   Solutions:
   We need to find the angle that the bearing A forms with the north. The 4 directions are given, and we know east is 90° from north. Subtract the given angle from 90°:
   $A = 90° - 40° = 50°$

   The three-point bearing is 050°.

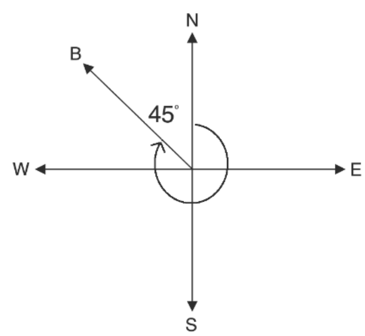
3. Find the three-point bearing of X:



Solution:
We know that south is 180° from north. Subtract the given angle from 180° to find the bearing of X from the north direction. $X = 180° - 32° = 148°$

The three-point bearing is 148°.

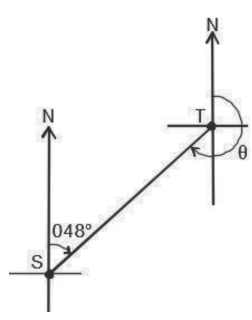
4. Find the three-point bearing of B:



Solution:
We know that a full revolution is 360°. Subtract the given angle from 360° to find the bearing of B from north. $B = 360° - 45° = 315°$

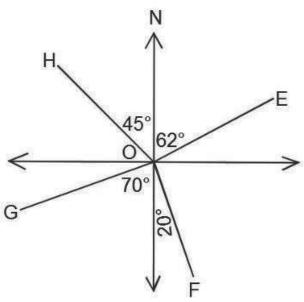
The three-point bearing is 315°.

5. If the bearing of T from S is 048°, find the bearing of S from T.
Solution:
First draw a diagram to visualise the problem:



Add 180° to find the reverse bearing, since the bearing from S to T is less than 180°:
$\theta = 48° + 180° = 228°$

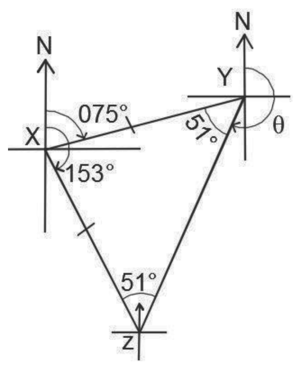
6. Find the three-figure bearings of the points E, F, G, and H from point O:



Solution:
Point E: $062°$
Point F: $180° - 20° = 160°$
Point G: $180° + 70° = 250°$
Point H: $360° - 45° = 315°$

7. If the bearing of Y from X is 281°, find the bearing of X from Y.
Solution:
Calculate the reverse bearing: $281° - 180° = 101°$

8. The bearing from X to Y is 075°, and the bearing from X to Z is 153°, where X, Y, and Z are 3 points on the plane. If X is equidistant from Y and Z, find the bearing from Y to Z.
Solution:
First, draw a diagram:



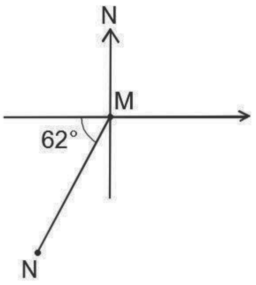
Find the equal angles of the isosceles triangle:

Angle of triangle at $X = 153° - 75° = 78°$
Solve for other angles: $180° - 78° = 102° \rightarrow 102° \div 2 = 51°$

Find the angle formed by line YZ and the south direction: $75° - 51° = 24°$
Bearing from Y to Z: $180° + 24° = 104°$

Practice

1. Draw diagrams to show the following bearings:
    a. 027°
    b. 151°
    c. 265°
2. If the bearing of X from Y is 200°, find the bearing of Y from X.
3. If the bearing from A to B is 012°, find the bearing from B to A.
4. In the diagram, find:
    a. The bearing of N from M
    b. The bearing of M from N



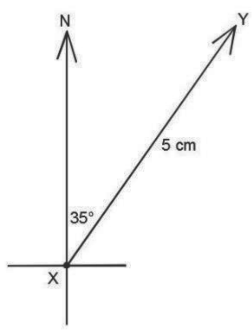
Lesson Title: Distance-bearing form	**Theme:** Bearings
Practice Activity: PHM2-L064	**Class:** SSS 4

Learning Outcomes
By the end of the lesson, you will be able to:
1. Write the distance and bearing of one point from another as (r, θ).
2. Interpret a distance-bearing problem and draw a corresponding diagram.

Overview

This lesson is on distance-bearing form. This is another way to describe bearings that use the distance between two points.

Consider the following example:

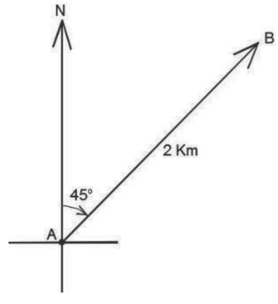


The bearing from X to Y can be written as $\overrightarrow{XY} = (5\text{ cm}, 035°)$. The distance and three-point bearing are given in brackets.

In general, the position of a point Q from another point P can be represented by $\overrightarrow{PQ} = (r, \theta)$, where r is the distance between the 2 points, and θ is the three-point bearing from P to Q.

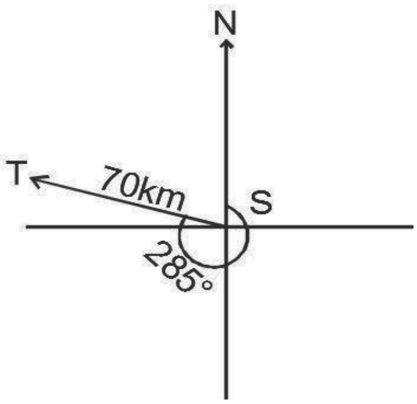
Solved Examples

1. A hunter starts at point A and travels through the bush 2 km in the direction 045° to point B. Give the distance and bearing, and draw a diagram.
   Solution:
   Distance $= 2$ km
   Bearing: $\overrightarrow{AB} = 045°$
   In distance-bearing form, this is $\overrightarrow{AB} = (2\text{ km}, 045°)$
   Diagram →



2. A boat leaves a port S and travels on the sea to another point T, 70 km in the direction $285°$. Draw the diagram and give the bearing of the ship in distance-bearing form.
Solution:

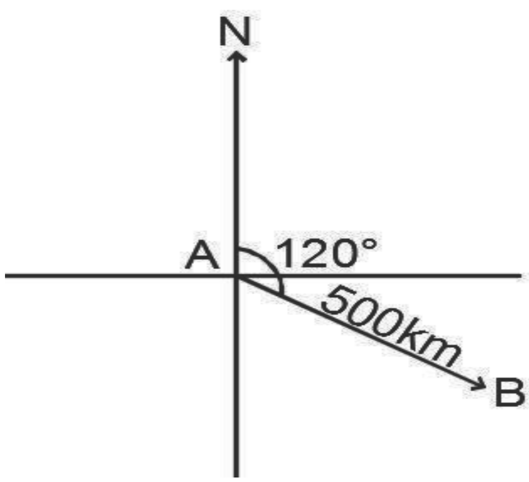
Diagram:



Distance $= 70\ \mathrm{km}$
Bearing: $\overrightarrow{ST} = 285°$
Distance-bearing form: $\overrightarrow{ST} = (70\ \mathrm{km}, 285°)$

3. A vehicle leaves a point A to another point B, 500 km in the direction $120°$. Draw the diagram and give the distance and three-point bearing.
Solution:

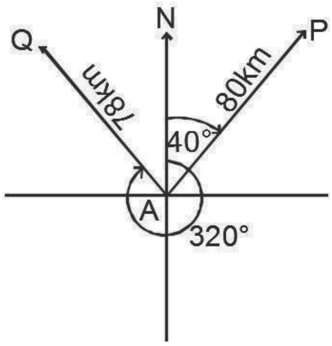
Diagram:



Distance $= 500\ \mathrm{km}$
Bearing: $\overrightarrow{AB} = 120°$

4. Two boats leave a port A. One travels 80 km in the direction $040°$ to a point P, the other travels 78 km in the direction $320°$ to a point Q. Draw the diagram and give the bearing for each in distance-bearing form.
Solution:

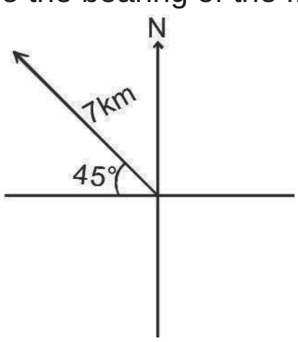
Diagram:



Bearing of point P: $\overrightarrow{AP} = (80 \text{ km}, 040°)$
Bearing of point Q: $\overrightarrow{AQ} = (78 \text{ km}, 320°)$

Practice

1. A vehicle starts at point M and travels 180 km in the direction 060° to point N. Draw the diagram and give the bearing of the vehicle in distance-bearing form.
2. A boat leaves a port P on sea and travels to another point Q, 50 km in the direction $210°$. Draw the diagram, and give the bearing of the boat in distance-bearing form.
3. Two boats leave a port P. One travels 4 km in the direction $320°$ to a point Q, the other travels 6 km in the direction $230°$ to a point R. Draw the diagram and give the distance and bearing of each.
4. Give the bearing of the figure below in distance-bearing form.



Lesson Title: Bearing problem solving – Part 1	**Theme:** Bearings
Practice Activity: PHM2-L065	**Class:** SSS 4

Learning Outcomes
By the end of the lesson, you will be able to:
1. Solve bearings problems with right triangles.
2. Apply Pythagoras' theorem and trigonometric ratios to calculate distance and direction.

Overview

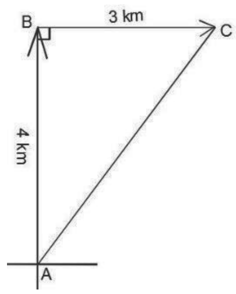
This lesson is on solving bearing problems. You will use Pythagoras' theorem and trigonometric ratios to solve for distance and direction.

Recall that when you encounter a bearings problem, the first step is to draw a diagram. At times, this diagram forms a right-angled triangle. You can apply Pythagoras' theorem and trigonometry to find missing sides and angles in such triangles.

Solved Examples

1. Hawa walked 4 km from point A to B in the north direction, then 3 km from point B to C in the east direction.
   a. How far is she from her original position?
   b. What is the bearing from A to C?

   Solutions:
   First, draw a diagram:



   Draw her movement in the north direction and the east direction. These two lengths can be connected to form a triangle, as shown above.
   a. Use Pythagoras' theorem to find the distance from C to A:

$\|AB\|^2 + \|BC\|^2$	$=$	$\|AC\|^2$	Apply Pythagoras' theorem
$4^2 + 3^2$	$=$	$\|AC\|^2$	Substitute known lengths
$16 + 9$	$=$	$\|AC\|^2$	Simplify
25	$=$	$\|AC\|^2$	

$$\sqrt{25} = \sqrt{|AC|^2}$$ Take the square root of both sides

$$5 \text{ km} = |AC|$$

She is 5 km from her original position.

b. Use trigonometry to find the angle of the bearing from A to C. We can choose any trigonometric function, because we know the lengths of all 3 sides. Let's use the tangent:

$\tan A = \frac{3}{4} = 0.75$ Apply the tangent ratio

$\tan^{-1}(\tan A) = \tan^{-1}(0.75)$ Take the inverse tangent of both sides

$A = \tan^{-1}(0.75)$

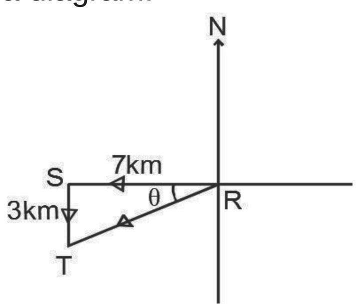
$A = 36.87°$ From the tangent table

The bearing from A to C is $\overrightarrow{AC} = (5 \text{ km}, 037°)$.

2. A farmer walks 7 km from his house, R, to a point S due west. He then walks 3 km due south from that point to his farm, T.
   a. How far is he from his house?
   b. What is the baring from his house to his farm?

Solutions:

First, draw a diagram:



a. Use the right-angled triangle RST to find the distance RT (which is the distance from the house to the farm. Use Pythagoras' theorem to find RT.

$$|RT|^2 = |RS|^2 + |ST|^2$$
$$= 7^2 + 3^2$$
$$= 58$$
$$RT = \sqrt{58}$$
$$= 7.6 \text{ km}$$

Therefore, the distance from the farmer's house to his farm is 7.6 km.

b. The bearing is the angle $180° + (90 - \theta)°$. The angle θ can be found from the triangle RST.

$$\tan\theta = \frac{3}{7}$$
$$= 0.4286$$
$$\theta = \tan^{-1} 0.4286$$

$$= \quad 23.2°$$

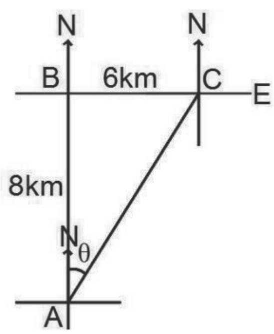
Find the bearing from the house to the farm:

$$180° + (90 - \theta)° = 180° + (90 - 23.2)° = 246.8°$$

The bearing from his house to his farm is $\overrightarrow{RT} = (7.6 \text{ km}, 247°)$.

3. Suppose a car leaves a point A and moves northwards for 8 kilometres to point B before turning right. It then moves eastward for 6 kilometres to point C. Represent the bearing of the car's displacement in distance bearing form.

Solution:



Use Pythagoras' theorem to find the distance C to A.

$\lvert AB\rvert^2 + \lvert BC\rvert^2$	$=$	$\lvert AC\rvert^2$	Apply Pythagoras' theorem
$8^2 + 6^2$	$=$	$\lvert AC\rvert^2$	Substitute the known lengths
$64 + 36$	$=$	$\lvert AC\rvert^2$	Simplify
100	$=$	$\lvert AC\rvert^2$	
$\sqrt{100}$	$=$	$\sqrt{\lvert AC\rvert^2}$	Take the square root of both sides
10 km	$=$	$\lvert AC\rvert$	

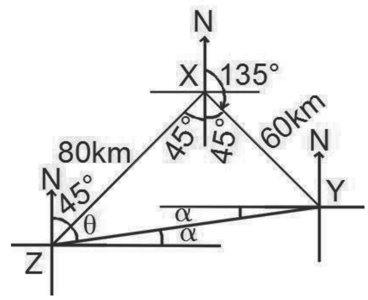
Use the tangent ratio to find θ:

$\tan\theta$	$=$	$\frac{6}{8}$	Apply the tangent ratio
$\tan\theta$	$=$	0.75	
$\tan^{-1}(\tan A)$	$=$	$\tan^{-1}(0.75)$	Take the inverse tangent of both sides
A	$=$	$\tan^{-1}(0.75)$	
A	$=$	$36.87°$	

The bearing from A to C is $\overrightarrow{AC} = (10 \text{ km}, 037°)$.

4. y is 60 km away from x on a bearing of $135°$, and z is 80 km away from x on a bearing of $225°$. Find the:
   a. Distance of z from y
   b. Bearing of z from y

   Solutions:



a. Note that the points X, Y and Z form a right-angled triangle, where X is a right angle. Use Pythagoras' theorem:

$\lvert ZY\rvert^2$	$=$	$\lvert XZ\rvert^2 + \lvert XY\rvert^2$	Apply Pythagoras' theorem
$\lvert ZY\rvert^2$	$=$	$80^2 + 60^2$	Substitute the known lengths
$\lvert ZY\rvert^2$	$=$	$6400 + 3600$	Simplify
$\lvert ZY\rvert^2$	$=$	$10{,}000$	
$\sqrt{\lvert ZY\rvert^2}$	$=$	$\sqrt{10{,}000}$	Take the square root of both sides
ZY	$=$	100 km	

b. From the bearing diagram, the bearing of z from y is $270° - \alpha$. First, find the angle labeled θ, and use it to find α and the bearing.

Use the tangent ratio to find θ:

$\tan\theta$	$=$	$\frac{60}{80} = \frac{3}{4}$	Apply the tangent ratio
$\tan\theta$	$=$	0.75	
θ	$=$	$\tan^{-1}(0.75)$	Take the inverse tangent of both sides
θ	$=$	$37°$	From the tangent table

Find α using the right angle at z: $\alpha = 90° - 45° + 37° = 8°$

Therefore, the bearing of z from y is: $270° - \alpha = 270° - 8° = 262°$

Practice

1. Abdul walked 3 km from his house to a point A in the north direction then 1 km to a school building B in the east direction.
   a. How far is Abdul from his house to the school?
   b. What is his bearing from his house to the school?
2. Isatu walked from a point P, 2 km in the west direction to a point Q and then 5 km in the north direction to a point M. Find Isatu's distance and bearing from her original position to the point M.
3. John walked 8 km in the west direction from a point O to a point M and then 5 km in the south direction to a point E. What is the distance and bearing from point O to point E?

4. A car leaves a point P and travels due south with a speed of 50 km/h to a point E in 1 hour and then travels due east with a speed of 20 km/h to a point M in a further 2 hours. Find the distance and bearing of the car from point P to point M.
5. A ship leaves port P and sails 15 km on a bearing of $045°$ to port Q. It then sails 20 km on a bearing of $135°$ to port R.
   a. Represent the information in a diagram.
   b. Calculate correct to the nearest whole number:
      i. The distance from P to R.
      ii. The bearing of R from P.
6. A village P is 10 km from a village Q, on a bearing $065°$. Another village R is 8 km from Q on a bearing of $155°$. Calculate:
   a. The distance of R from P to the nearest kilometre.
   b. The bearing of R from P, to the nearest degree.

Lesson Title: Bearing problem solving – Part 2	**Theme:** Bearings
Practice Activity: PHM2-L066	**Class:** SSS 4

Learning Outcomes

By the end of the lesson, you will be able to:

1. Solve bearings problems with acute and obtuse triangles.
2. Apply the sine and cosine rules to calculate distance and direction.

Overview

This lesson is on solving bearings problems. You will use the sine and cosine rules to solve for distance and direction.

Recall that the sine and cosine rules can be used to find missing sides and angles in acute and obtuse triangles. These may be applied to bearings problems that feature such triangles in their diagrams. Recall the information that is needed to apply either the sine rule or cosine rule:

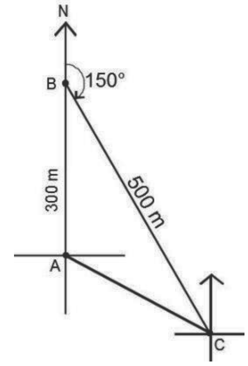
- **Sine Rule:** Use if given any 2 angles and 1 side, or 2 sides and the angle opposite 1 of them.
- **Cosine Rule:** Use if given two sides and the angle between them.

Solved Examples

1. A man walks 300 metres due north, then 500 metres at a bearing of 150°.
   a. How far is he from his original location?
   b. What is his bearing from his original position?

   Solutions:

   First, draw the diagram. →



   a. Find the angle inside the triangle at B:

$$B = 180° - 150° = 30°$$

   Apply the cosine rule to find his distance:

$\|AC\|^2$	$=$	$\|AB\|^2 + \|BC\|^2 - 2\|AB\|\|BC\| \cos B$	Formula
	$=$	$300^2 + 500^2 - 2(300)(500) \cos 30°$	Substitute values from the triangle
	$=$	$90{,}000 + 250{,}000 - 300{,}000 \cos 30°$	Simplify
	$=$	$340{,}000 - 300{,}000(0.8660)$	Substitute $\cos 30° = 0.8660$
	$=$	$340{,}000 - 259{,}800$	
$\|AC\|^2$	$=$	$80{,}200$	

$|AC| \quad = \quad \sqrt{80{,}200} = 283.2$ m to 2 d.p. Take the square root of both sides

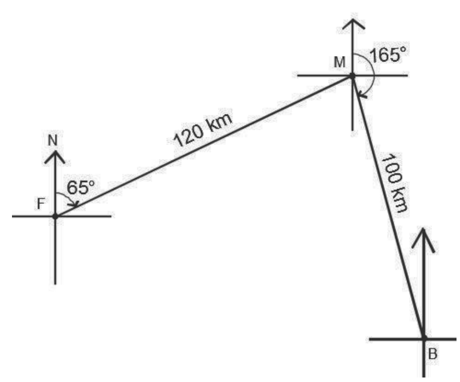
b. To find the bearing of C from A, find angle BAC.
Note that the bearing of C from A is $180° - (\angle ABC + \angle BCA)$, using the interior angles of the triangle ABC.
Use the sine rule to find angle $\angle BCA$:

$$\frac{|AC|}{\sin B} = \frac{300}{\sin C}$$
$$\frac{283.2}{\sin 30} = \frac{300}{\sin C}$$
$$\sin C = \frac{300 \times \sin 30°}{283.2}$$
$$\sin C = \frac{300 \times 0.5}{283.2}$$
$$\sin C = 0.5297$$
$$\angle C = 32°$$

Bearing of C from A: $180° - (\angle ABC + \angle BCA) = 180° - (30 + 32)° = 118°$

2. An airplane flies from Freetown (point F) on a bearing of 65° to Makeni (point M), a distance of 120 km. It arrives in Makeni, but changes course and flies to Bo, a distance of 100 km, on a bearing of 165°.
   a. What is the distance from Freetown to Bo?
   b. What is the bearing of Bo from Freetown?

Solutions:
First, draw the diagram. →



a.
Note that we can use the cosine rule if we find the measure of the triangle at angle M.
Step 1. Find the angle at M.
- Note that there are opposite interior angles with point F, which has a known angle of $65°$.
- The unknown angle outside of the triangle at M is $180° - 65° = 115°$
- Subtract the known angles at M from 360: $360° - 115° - 165° = 80°$

Step 2. Apply the cosine rule:

$\lvert FB\rvert^2$	$=$	$\lvert FM\rvert^2 + \lvert MB\rvert^2 - 2\lvert FM\rvert\lvert MB\rvert \cos M$	Formula
	$=$	$120^2 + 100^2 - 2(120)(100) \cos 80°$	Substitute values from the triangle
	$=$	$14{,}400 + 10{,}000 - 24{,}000 \cos 80°$	
	$=$	$24{,}400 - 24{,}000\ (0.1736)$	Substitute $\cos 80° = 0.1736$

$$= 24{,}400 - 4{,}166.4$$

$$|FB|^2 = 20{,}233.6$$

$|FB| = \sqrt{20{,}233.6} = 142.24$ km to 2 d.p. Take the square root of both sides

The distance from Freetown to Bo is 142.24 kilometres.

b. Note that to find the bearing, we can use the sine rule to find the missing angle of F inside the triangle. We will add this to 65°, the known angle at F.

2. $\frac{100}{\sin F} = \frac{142.24}{\sin 80°}$ Substitute in the formula

$\sin F = \frac{100 \sin 80°}{142.24}$ Solve for F

$\sin F = \frac{100 \times 0.9848}{142.24}$

$\sin F = 0.6024$

$F = \sin^{-1} 0.6024$ Take the inverse sine of both sides

$F = 37.04°$ Use the sine table

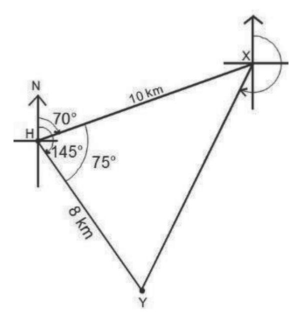
Add: Bearing $= 37.04° + 65° = 102.04°$

The bearing from Freetown to Bo is $\overrightarrow{FB} = (142.24 \text{ km}, 102°)$

3. Village X is 10 km from the nearest hospital on a bearing of 70°. Village Y is 8 km from the same hospital on a bearing of 145°. Calculate:
   a. The distance of village Y from village X, to the nearest kilometre.
   b. The bearing of Y from X, to the nearest degree.

Solutions:

First draw the diagram. →



a. Calculate distance:

Step 1. Find the angle of H in the triangle:

$$145° - 70° = 75°$$

Step 2. Use the cosine rule to find $|XY|$:

$|XY|^2 = |HX|^2 + |HY|^2 - 2|HX||HY| \cos H$ Formula

$= 10^2 + 8^2 - 2(10)(8) \cos 75°$ Substitute values from triangle

$= 100 + 64 - 160 \cos 75°$

$= 164 - 160\,(0.2588)$ Substitute $\cos 75° = 0.2588$

$= 164 - 41.408$

$|XY|^2 = 122.592$

$|XY| = \sqrt{122.592} = 11$ km Take the square root

b. To find the bearing of Y from X, first find the other angles at X:

Find the angle **inside** the triangle at X using the sine rule:

$\frac{8}{\sin X} = \frac{11}{\sin 75°}$ Substitute in the formula

$$\begin{aligned}
\sin X &= \frac{8\sin 75°}{11} && \text{Solve for } X \\
\sin X &= \frac{8\times 0.9659}{11} \\
\sin X &= 0.7025 \\
X &= \sin^{-1} 0.7025 && \text{Take the inverse sine of both sides} \\
X &= 44.63° && \text{Use the sine table}
\end{aligned}$$

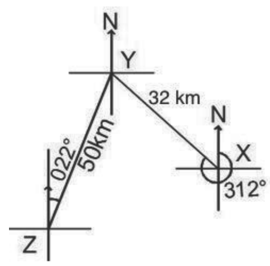
Find the small angle at X that is part of the bearing. Subtract $44.63°$ from $70°$, which is an opposite interior angle of the $70°$ angle at H: $70° - 44.63° = 25.37°$.

Add to find the full bearing: $180° + 25.37° = 205.37°$

The bearing to the nearest degree is $205°$.

Practice

1. Y is 60 km away from X on a bearing of $135°$. Z is 80 km away from X on a bearing of $225°$. Find the:
    a. Distance of Z from Y
    b. Bearing of Z from Y
2. The diagram shows the positions of three points X, Y and Z on a plane. The bearing of Y from X is $312°$ and that of Y from Z is $022°$. If $|XY| = 32$ km and $|ZY| = 50$ km. Calculate, correct to one decimal place:
    a. $|XZ|$
    b. The bearing of Z from X



3. A ship travels 3 km from a port P on a bearing of $080°$ and then 4 km on a bearing $047°$. Find its distance and bearing from P.
4. A bus moves from Moyamba on a bearing of $75°$ to Massiaka, a distance of 130 km. It arrives in Massiaka, but changes course and moves to Kono, a distance of 90 km, on a bearing of $160°$.
    a. Find the distance from Moyamba to Kono, to the nearest kilometre.
    b. Find the bearing of Kono from Moyamba, to the nearest degree.
5. John's house is 20 km from his school on a bearing $80°$. Kadie's house is 12 km from the same school on a bearing of $155°$. Calculate the distance of John's house from Kadie's house to the nearest kilometre.
6. A woman walks due east from point A to point B, a distance of 8 kilometres. She then changes direction and walks 6 km to point C on a bearing of 048°.
    a. What is the distance from A to C?
    b. What is the bearing of C from A?

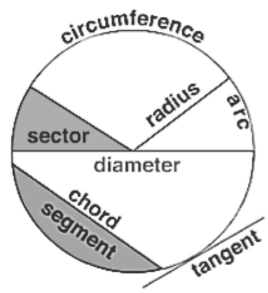
Lesson Title: Circles	Theme: Geometry
Practice Activity: PHM2-L067	**Class:** SSS 4

Learning Outcomes
By the end of the lesson, you will be able to:
1. Calculate the circumference and area of a circle.
2. Calculate the length of an arc and area of a sector of a circle.

Overview

This lesson is a review lesson on calculating the circumference and area of a circle. From the circumference and area, we are also able to calculate the length of an arc and area of a sector.

In order to perform calculations on circles, you must have an understanding of its parts. The parts of a circle are shown in the diagram and table below. These will be used in the following lessons.

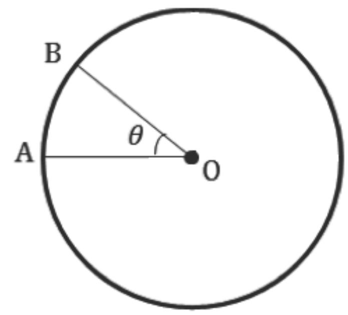


Parts of circle	Description
Circumference	The distance around a circle.
Radius	The distance from the centre of the circle to any point on the circumference. It is also half of the diameter.
Diameter	A straight line passing through the centre of the circle, touching both sides of the circumference. It is twice the length of the radius.
Chord	A straight line joining two points on the circumference of a circle. The diameter is a special kind of chord which passes through the centre of the circle.
Arc	A section of the circumference of a circle.
Sector	A section of a circle bounded by two radii and an arc.
Segment	A section of a circle bounded by a chord and an arc.
Tangent	A straight line touching the circumference at a given point.

Semi-circle	Half of a circle.
Quadrant	Quarter of a circle.

The **circumference** of a circle is given by the formula $C = 2\pi r$ where r is the radius of the circle. The **area** of a circle is given by the formula $A = \pi r^2$ where r is the radius of the circle. π is a constant, and we use estimated values of 3.14 or $\frac{22}{7}$ for π.

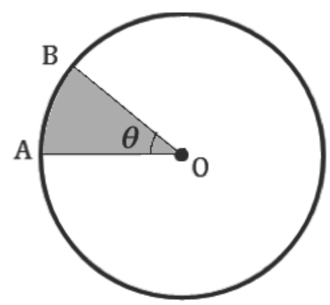
An **arc** is a part of the circumference of a circle.



The circle to the right shows an arc AB subtended by an angle θ at the centre, O, of a circle. We know that the angle subtended by the circumference of a circle is $360°$ (a full revolution). The lengths of any arcs of the circle are in proportion to the angles they subtend. In the diagram, the length of arc AB is proportional to θ. To find the length of the arc, multiply the circumference by θ as a fraction of $360°$.

$$\text{Length of arc} = \frac{\theta}{360°} \times C = \frac{\theta}{360°} \times 2\pi r$$

Similarly, a **sector** is part of the area of a circle.



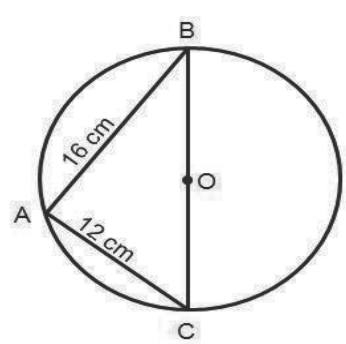
The area of a sector of a circle is proportional to the angle of the sector. The circle to the right shows sector OAB subtended by the angle θ at the centre, O, of a circle. An entire circle is a sector with an angle of $360°$ and area πr^2. All other sectors have areas in proportion to the angle of the sector.

$$\text{Area of sector} = \frac{\theta}{360°} \times A = \frac{\theta}{360°} \times \pi r^2$$

Solved Examples

1. The radius of a circle is 7 cm. Using $\pi = \frac{22}{7}$, find:
   a. The length of its diameter.
   b. The length of its circumference.
   c. The area of the circle.

   Solutions:
   a. Diameter is twice radius: $d = 2r = 2(7) = 14$ cm.
   b. Circumference: $C = 2\pi r = 2 \times \frac{22}{7} \times 7 = \frac{308}{7} = 44$ cm
   c. Area: $A = \pi r^2 = \frac{22}{7} \times 7^2 = 22 \times 7 = 154$ cm^2



2. In the diagram, triangle ABC is cut out from the circle with centre O. If $|AB| = 16\ cm$ and $|AC| = 12$ cm, find the area of the remaining part of the circle. (Use $\pi = 3.14$)

Solution:

To find the area of the remaining part, find the area of the triangle and subtract from the area of the circle. ΔABC is a right angled triangle since it is incribed in a semi-circle.

Triangle Area $= \frac{1}{2}|\mathrm{AB}| \times |\mathrm{AC}|$

$= \frac{1}{2} \times 16 \times 12$ Substitute the given sides

$= 96\ \mathrm{cm}^2$

To find the area of the circle, you need the radius, which is $\frac{1}{2}$|BC|. Find |BC|

Using Pythagoras' theorem:

$|BC|^2 = |\mathrm{AB}|^2 + |\mathrm{AC}|^2$

$= 16^2 + 12^2$ Substitute the given sides

$|BC|^2 = 400$

|BC| $= 20$ Take the square root

Circle Area $= \pi r^2$

$= 3.14 \times (10)^2$ Substitute $r = \frac{1}{2}|\mathrm{BC}| = \frac{20}{2} = 10$

$= 314\ \mathrm{cm}^2$

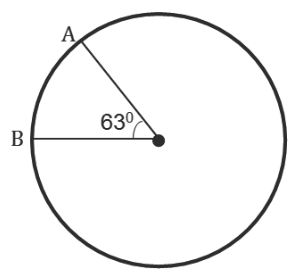
Area of remaining part of the circle = Area of circle – Area of triangle:

Area of remaining part $= 314 - 96 = 218\ \mathrm{cm}^2$

3. An arc subtends an angle of 63° at the centre of a circle of radius 12 cm. Find the length of the arc. [Use $\pi = \frac{22}{7}$]

Solution:

First draw a diagram of the problem (shown below).



$|\mathrm{AB}| = \frac{\theta}{360} \times 2\pi r$

$= \frac{63}{360} \times 2 \times \frac{22}{7} \times 12$ Substitute $\theta = 63, r = 12$

$= \frac{63 \times 2 \times 22 \times 12}{360 \times 7}$ Simplify

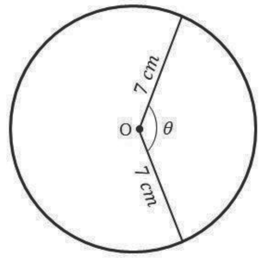
$|\mathrm{AB}| =$ 13.2 cm

4. An arc of a circle of radius 7 cm is 14 cm long. What angle does the arc subtend at the centre of the circle? Give your answer to 1 decimal place. Use $\pi = \frac{22}{7}$

Solution:

Length of arc $= \frac{\theta}{360^0} \times 2\pi r$

$$
\begin{aligned}
14cm &= \frac{\theta}{360^0} \times 2 \times \frac{22}{7} \times 7cm \\
\frac{14cm}{1} &= \frac{\theta \times 2 \times 22 \times 7}{360^0 \times 7} \\
1 \times \theta \times 2 \times 22 \times 7 &= 360^0 \times 7 \times 14 \\
\theta \times 308 &= 35{,}280^0 \\
\theta &= \frac{35{,}280^0}{308} \\
\theta &= 114.5^0
\end{aligned}
$$

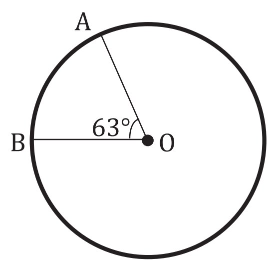


5. An arc subtends an angle of 63° at the centre of a circle of radius 12 cm. Find the area of the sector correct to the nearest cm². Use $\pi = 3.14$.

   Solution:

   First draw a diagram of the problem (shown below).

$$
\begin{aligned}
A &= \frac{\theta}{360} \times \pi r^2 \\
&= \frac{63}{360} \times 3.14 \times 12^2 && \text{Substitute } \theta = 63°, r = 12 \\
&= 79.128 && \text{Simplify} \\
A &= 79 \text{ cm}^2
\end{aligned}
$$



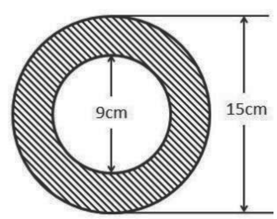
The area of the sector to the nearest cm² = 79 cm².

Practice

1. Complete the table below. Use $\pi = \frac{22}{7}$.

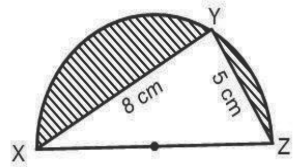
No.	Radius	Diameter	Circumference	Area
a.	12 m			
b.		42 m		
c.			616 mm	
d.		126 mm		

2. Mary's garden is circular in shape; its diameter is 7 m. If she plants rose trees 50 cm apart round the edge of the garden, how many rose trees does she need? Use $\pi = \frac{22}{7}$.
3. The area of a circular track is 616 m^2. Find the radius of the track. Use $\pi = \frac{22}{7}$.
4. Find the area of the shaded portion in the figure shown. Give your answer to the nearest whole number. [Use $\pi = 3.14$]



5. Find the length of an arc of radius 12 cm if it subtends an angle of 120° at its centre. Give your answer to 2 significant figures. Use $\pi = \frac{22}{7}$.

6. The angle of a sector of a circle of diameter 23 cm is $150°$. Find the area of the sector, to the nearest whole number. Use $\pi = 3.14$.
7. A sector of a circle of radius 13 cm has an area 64.6 cm^2. Calculate the angle of the sector, correct to the nearest degree. Use $\pi = 3.14$.
8. The area of the sector of a circle is 114 cm^2. If the angle of the sector is $108°$. Find the radius of the circle to the nearest whole number. Use $\pi = \frac{22}{7}$.
9. In the diagram, XYZ is a semi-circle. If |XY|= 8 cm and |YZ|= 5 cm, calculate correct to 3 significant figures:
   a. The radius of the circle.
   b. The area of the shaded part ($\pi = 3.142$).



Lesson Title: Circle Theorems 1 and 2	**Theme:** Geometry
Practice Activity: PHM4-L068	**Class:** SSS 4

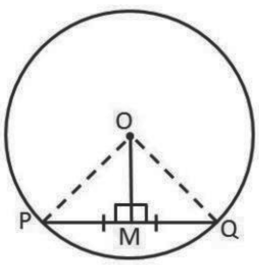
Learning Outcomes
By the end of the lesson, you will be able to:
1. Solve problems related to the perpendicular bisector of a chord.
2. Solve problems related to angles subtended at the centre or circumference of a circle.

Overview

In Geometry, rules or **theorems** are used to determine lengths and angles in circles. These theorems are true for every circle regardless of the size of the circle. We have numbered the theorems in the order we will be working with them. They may be numbered differently in other textbooks.

The first theorem deals with the perpendicular bisector of a chord from the centre of a circle.

Circle Theorem 1: A straight line from the centre of a circle that bisects a chord is at right angles to the chord



In the diagram, we have a circle with centre O and line OM to mid-point M on chord PQ such that $|PM| = |QM|$. Note that OM $\perp$ PQ. The $\perp$ symbol means "is perpendicular" to the chord.

Circle Theorem 2: The angle subtended at the centre of a circle is twice that subtended at the remaining part of the circumference.

For each of the circles below:
- Arc AB subtends $\angle$ AOB at the centre of the circle.
- Depending on the position of point P on the circle, we can have 3 different ways of how $\angle$ APB is formed at the circumference of the circle.
- The 3 ways are shown on the circles below.

In any circle below with centre O, arc AB is subtending $\angle$AOB at the centre of the circle, and $\angle$APB at the circumference. In each circle, the angle subtended at the centre is twice that subtended at the circumference. In other words: $\angle AOB = 2 \times \angle APB$

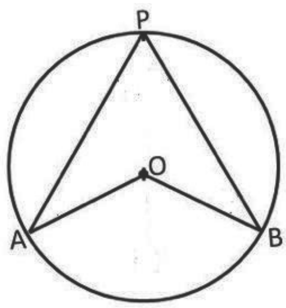


Figure a

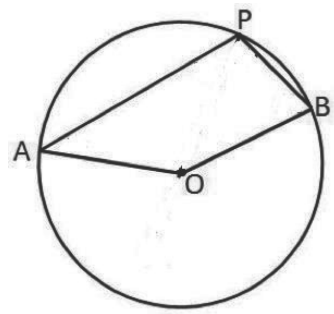


Figure b

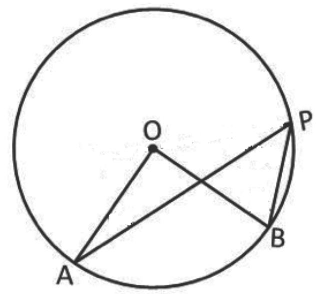


Figure c

Solved Examples

1. The radius of a circle is 12 cm. The length of a chord of the circle is 18 cm. Calculate the distance of the mid-point of the chord from the centre of the circle. Give your answer to the nearest cm.
   Solution:
   Using the circle from the proof:

M	=	mid-point of PQ	
|MQ|	=	9 cm	$\frac{1}{2}\times$|PQ| $= \frac{18}{2} = 9$
$\angle$OMQ	=	$90°$	line from centre to mid-point $\perp$
|OQ|^2	=	|OM|^2 + |MQ|^2	Pythagoras Theorem
12^2	=	|OM|$^2 + 9^2$	substitute|OQ| $= 12$ cm, |MQ| = 9 cm
|OM|^2	=	$12^2 - 9^2$	
	=	$144 - 81$	
|OM|	=	$\sqrt{63} = 7.94$	
|OM|	=	8 cm	

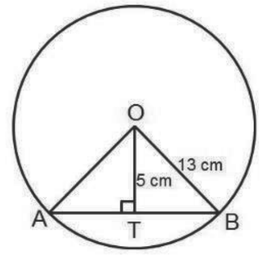
   The distance from the mid-point of the chord to the centre of the circle is 8 cm to the nearest cm.

2. A chord is 5 cm from the centre of a circle of diameter 26 cm. Find the length of the chord.
   Solution:

$$r = \frac{26}{2}$$
$$r = 13 \text{ cm}$$

From $\triangle$OBT

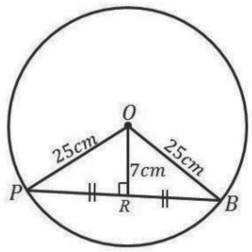
$$\begin{aligned} OB^2 &= OT^2 + BT^2 \\ 13^2 &= 5^2 + BT^2 \\ 169 &= 25 + BT^2 \\ 169 - 25 &= BT^2 \\ 144 &= BT^2 \\ BT &= 12 \text{ cm} \end{aligned}$$



$$\begin{aligned} \text{Length of chord AB} &= 2BT \\ &= 2 \times 12 \\ &= 24 \text{ cm} \end{aligned}$$

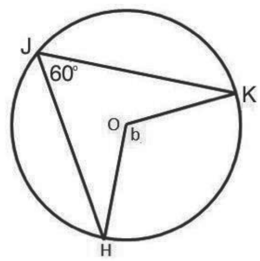
3. Find the length of the chord, which is at a distance of 7 cm from the centre of a circle, whose radius is 25 cm.

Solution:



$$\begin{aligned} \text{Radius} &= 25 \text{ cm} \\ \text{From } \triangle\text{OBR} & \\ OB^2 &= BR^2 + OR^2 \\ 25^2 &= BR^2 + 7^2 \\ 625 &= BR^2 + 49 \\ 625 - 49 &= BR^2 \\ 576 &= BR^2 \\ BR &= 24 \text{ cm} \\ \text{Length of chord PB} &= 2BR \\ &= 2 \times 24 \text{ cm} \\ &= 48 \text{ cm} \end{aligned}$$

4. Find the value of the unknown angle, *b*, in the diagram.

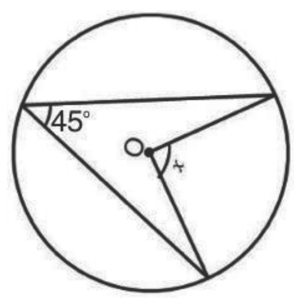


Solution:

Multiply the angle at the circumference by 2 to find the angle at the centre:

$$\begin{aligned} b &= 2 \times 60° \\ b &= 120° \end{aligned}$$

5. Find the value of $\angle x$ in the diagram:



Solution:

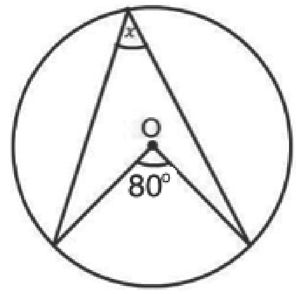
Multiply the angle at the circumference by 2 to find the angle at the centre:

$$\begin{aligned} x &= 2 \times 45^\circ \\ x &= 90^\circ \end{aligned}$$

6. Find the value of $\angle x$ in the diagram:

   Solution:

$$2x = 80^\circ$$
$$\frac{2x}{2} = \frac{80^\circ}{2}$$
$$x = 40^\circ$$

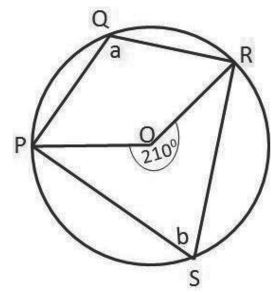


7. Given O is the centre of the circle, find the unknown angles a and b in the circle shown.

   Solution:

   Step 1. Assess and extract the given information from the problem.

   Reflex $\angle POR = 210°$



   Step 2. Use theorems and the given information to find all equal angles and sides on the diagram.

$a = \frac{1}{2} \times 210$ $\angle$at centre = 2$\angle$ at circumference

$a = 105°$

Obtuse $\angle POR = 360 - 210$ sum of $\angle$s in a circle $= 360°$

$= 150°$

   Step 3. Solve for b.

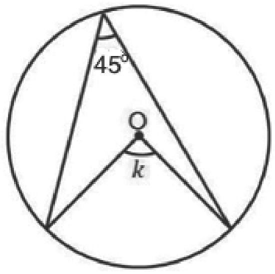
$b = \frac{1}{2} \times 150$ $\angle$at centre = 2$\angle$ at circumference

$b = 75°$

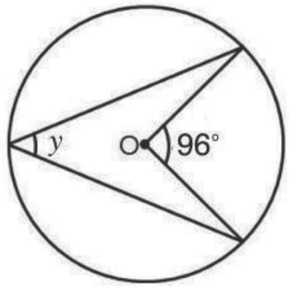
Practice

1. A chord of length 30 cm is 8 cm away from the centre of the circle. What is the radius of the circle?
2. A chord in a circle is 16 cm long and its perpendicular distance from the centre is 6 cm. What is the radius of the circle?
3. A circle has a radius of 1.3 cm. A chord has a length of 2.4 cm. Find the distance of this chord from the centre.
4. Find the length of the chord, which is at a distance of 17 cm from the centre of a circle, whose radius is 23 cm correct to 2 places of decimals.
5. Find the angles marked with letters in the following figures. In each figure, the letter O represents the centre of the circle.

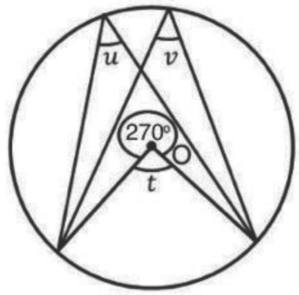
a.



b.

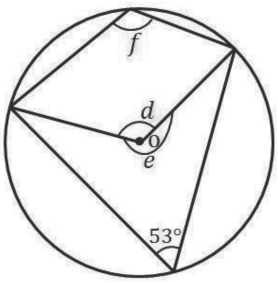


c.

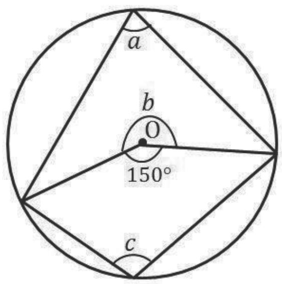


6. Find the angles marked with letters in the following figures. In each figure, the letter O represents the centre of the circle.

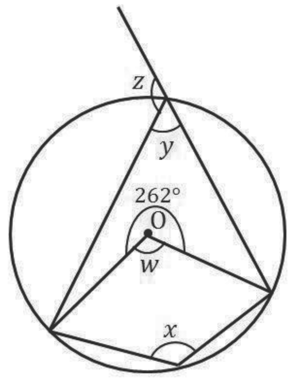
a.



b.



c.



Lesson Title: Circle Theorems 3, 4 and 5	**Theme:** Geometry
Practice Activity: PHM2-L069	**Class:** SSS 4

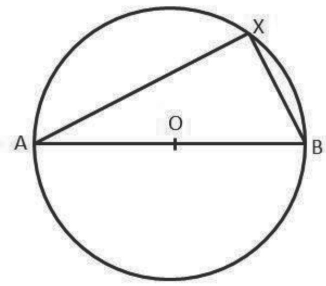
Learning Outcomes
By the end of the lesson, you will be able to:
1. Solve problems related to the angle in a semi-circle.
2. Solve problems related to angles in the same segment.
3. Solve problems related to opposite angles of a cyclic quadrilateral.

Overview

This lesson deals with 3 theorems, which are related to angles inside circles.

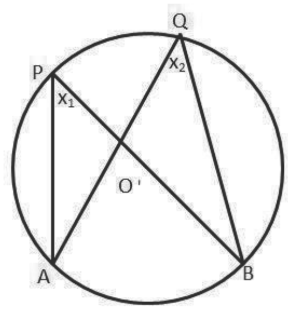
Circle Theorem 3: The angle in a semi-circle is a right angle.

In the diagram, we have a circle with centre O and diameter AB. X is any point on the circumference of the circle. For any such point X, $\angle AXB$=90°. This theorem shows that the angle of the diameter of a circle subtends a right angle at the circumference.



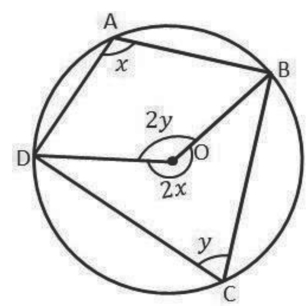
Circle Theorem 4: Angles in the same segment are equal.

In the diagram, we have circle with centre O with points P and Q on the circumference of the circle. Arc AB subtends $\angle APB$ and $\angle AQB$ in the same segment of the circle. Two angles subtended by the same arc are equal**:** $\angle APB$=$\angle AQB$.



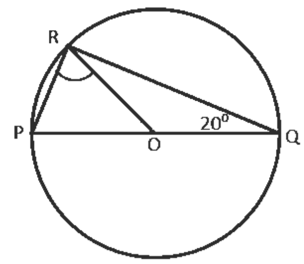
Circle Theorem 5: The opposite angles of a cyclic quadrilateral are supplementary.

A cyclic quadrilateral is a quadrilateral with all 4 vertices on the circumference of the circle. Both sets of opposite angles are supplementary (they sum to 180°). In the diagram, $\angle BAD + \angle BCD = 180°$ and $\angle ABC + \angle ADC = 180°$.



From this, it follows that the exterior angle of a cyclic quadrilateral is equal to the opposite interior angle. See Solved Example 7 for an example of this.

Solved Examples



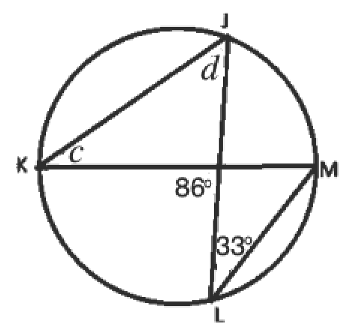
1. In the diagram at right, P, Q and R are points on a circle, with centre O. PQ is the diameter. If $\angle RQO$=20°, what is the size of $\angle PRO$

   Solution:

   Note that $\angle QRO = \angle RQO$ =20° because these are the base angles of isosceles triangle ΔQRO, which is formed by two radii.

   Note that $\angle PRQ = 90°$ because it subtends at the circumference. Subtract to find $\angle PRO$:

$$\begin{aligned}\angle PRO &= \angle PRQ - \angle QRO \\ \angle PRO &= 90 - 20 \\ \angle PRO &= 70°\end{aligned}$$



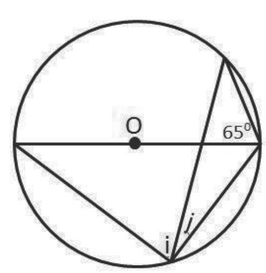
2. Find the measures of angles c and d in the diagram at right:

   Solution:

   Notice that c is subtended by the same arc (JM) as the $33°$ angle. Applying theorem 4, $\angle c = 33°$.

   Find angle d using the triangle that contains c and d. The other angle is $180° - 86° = 94°$.

$$\angle d = 180° - 33° - 94° = 53°$$



3. Given O is the centre of the circle at right, find the unknown angle in the circle shown.

   Solution:

   Step 1. Assess and extract the given information from the problem.

   Given angle $= 65°$

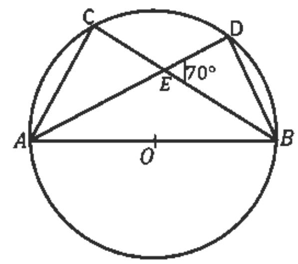
   Step 2. Use theorems and the given information to find all equal angles and sides on the diagram.

i	$=$	$65°$	$\angle$s in the same segment
$i + j$	$=$	90	$\angle$ in a semi-circle
$65 + j$	$=$	90	

   Step 3. Solve for j.

$$\begin{aligned}j &= 90 - 65 \\ j &= 25°\end{aligned}$$

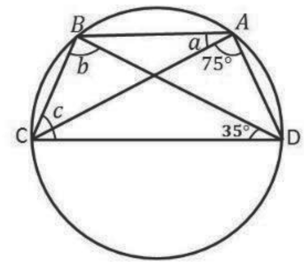
4. AB is the diameter of a circle at right, with the centre O. If $\angle DEB = 70^{\circ}$, find $\angle DBE$ and $\angle DAC$



Solution:

$$\begin{aligned} \angle BDE &= 90^{\circ} && \angle \text{ in a semi-circle} \\ \angle DEB + \angle BDE + \angle DBE &= 180^{\circ} && \angle\text{s in a triangle} \\ 70^{\circ} + 90^{\circ} + \angle DBE &= 180^{\circ} \\ 160^{\circ} + \angle DBE &= 180^{\circ} \\ \angle DBE &= 180^{\circ} - 160^{\circ} \\ \angle DBE &= 20^{\circ} \\ \angle DAC &= \angle DBE && \angle\text{s in the same segment} \\ \angle DAC &= 20^{\circ} \end{aligned}$$

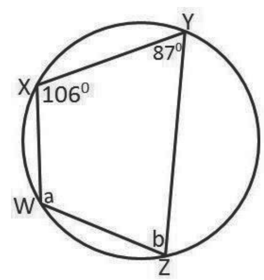
5. Find angles a, b, and c at right:



Solutions:

$$\begin{aligned} a &= 35^{\circ} && \angle\text{s in the same segment} \\ b &= 75^{\circ} && \angle\text{s in the same segment} \\ b + 35^{\circ} + c &= 180^{\circ} && \text{sum of the interior } \angle\text{s of a } \Delta \\ 75^{\circ} + 35^{\circ} + c &= 180^{\circ} \\ 110^{\circ} + c &= 180^{\circ} \\ c &= 180^{\circ} - 110^{\circ} \\ c &= 70^{\circ} \end{aligned}$$

6. Find the unknown angle in the circle shown at right:

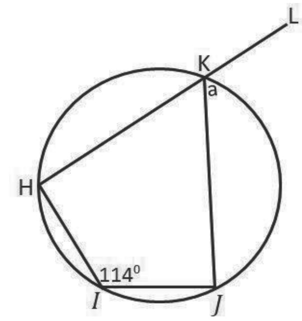


Solution:

Use theorem 5 and the given information to find all equal angles on the diagram.

$$\begin{aligned} a + 87 &= 180 && \text{opposite } \angle\text{s of cyclic quadrilateral} \\ a &= 180 - 87 \\ a &= 93^{\circ} \\ b + 106 &= 180 && \text{opposite } \angle\text{s of cyclic quadrilateral} \\ b &= 180 - 106 \\ b &= 74^{\circ} \end{aligned}$$

7. Find the unknown angle in the circle shown at right:



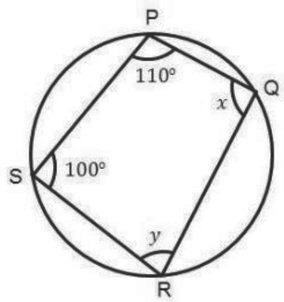
Solution:

Use theorem 5, the exterior angle of a cyclic quadrilateral is equal to the opposite interior angle.

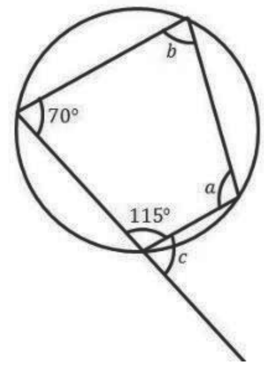
$$a = \angle HIJ = 114°$$

8. Find the angles marked with letters in the following diagrams:

a.



b.



Solutions:

Apply theorem 5.

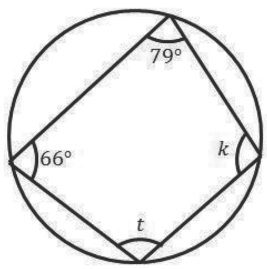
a.
$$\begin{aligned} x + 100^\circ &= 180^\circ \\ x &= 180^\circ - 100^\circ \\ x &= 80^\circ \\ y + 110^\circ &= 180^\circ \\ y &= 180^\circ - 110^\circ \\ y &= 70^\circ \end{aligned}$$

b.
$$\begin{aligned} a + 70^\circ &= 180^\circ \\ a &= 180^\circ - 70^\circ \\ a &= 110^\circ \\ b + 115^\circ &= 180^\circ \\ b &= 180^\circ - 115^\circ \\ b &= 65^\circ \\ c &= b \\ c &= 65^\circ \end{aligned}$$

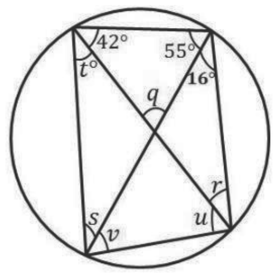
Practice

1. Find the angles marked with letters in the following diagrams below. O represents the centre of circles d., e. and f.

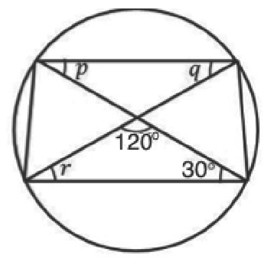
a.



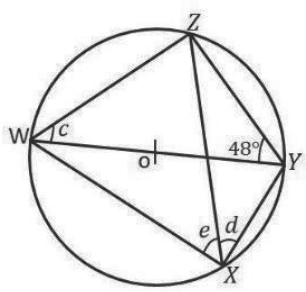
b.



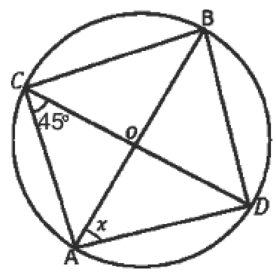
c.



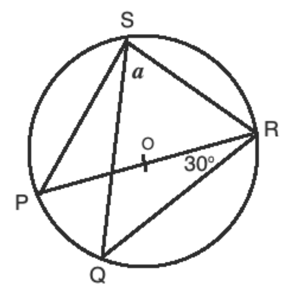
d.



e.



f.

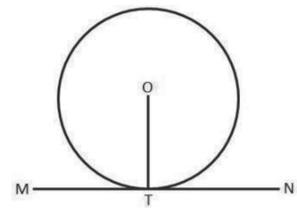


Lesson Title: Circle Theorems 6 and 7	**Theme:** Geometry
Practice Activity: PHM2-L070	**Class:** SSS 4

Learning Outcomes
By the end of the lesson, you will be able to:
1. Identify and draw the tangent line to a circle.
2. Solve problems related to the tangent to a circle.

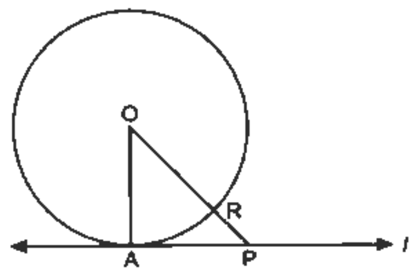
Overview

This lesson deals with the theorems concerning tangents to a circle.



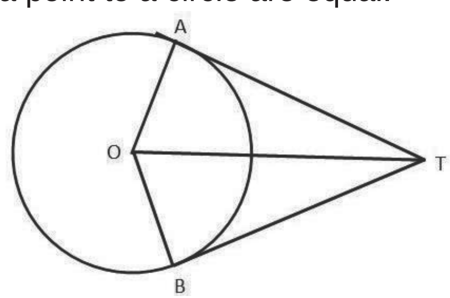
A tangent is a line which touches a circle at one point without cutting across the circle. It makes contact with a circle at only one point on the circumference. The line MN shown at right is a tangent to the circle with the centre O. It touches the circle at point T.

Circle Theorem 6: The angle between a tangent and a radius is equal to $90°$.



The shortest line from the centre of a circle to a tangent is a line that is perpendicular to the tangent. In the diagram, $\angle \text{OAP} = 90°$. We can write: radius $\perp$ tangent. Recall that $\perp$ means "is perpendicular to". In the diagram, OA $\perp$ l.

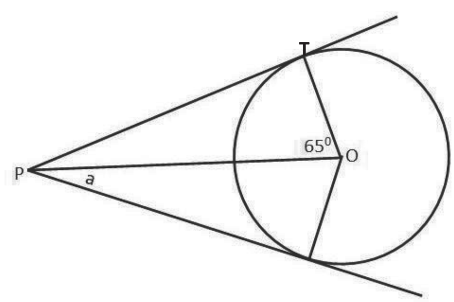
Circle Theorem 7: The lengths of the two tangents from a point to a circle are equal.



For a point T outside a circle with the centre O, TA and TB are tangents to the circle at A and B respectively. The lengths of TA and TB are equal. $|\text{TA}| = |\text{TB}|$

Since $\angle$AOT=$\angle$BOT and $\angle$ATO=$\angle$BTO, it also means that line TO bisects the angles at O and T. TO is therefore the line of symmetry for the diagram.

Solved Examples



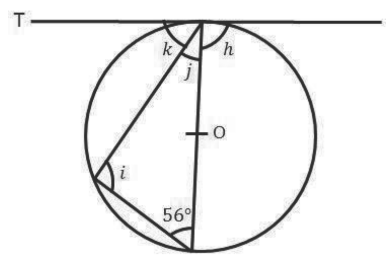
1. Find the missing angle in the given circle with centre O.

Solution:

Use theorems and the given information to find a.

$\angle PTO$	$=$	$90°$	radius $\perp$ tangent
a	$=$	$\angle TPO$	symmetry (equal tangents)
$a + 65 + 90$	$=$	180	$\angle$s in a triangle
a	$=$	$180 - 65 - 90$	
a	$=$	$25°$	

2. In the given figure, a line drawn through T is a tangent and O is the centre of the circle. Find the lettered angles.



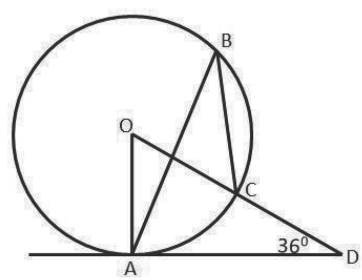
Solution:

$$i = 90^\circ \quad \angle \text{ in a semicircle}$$
$$i + j + 56^\circ = 180^\circ \quad \text{sum of interior } \angle\text{s in a } \Delta$$
$$90^\circ + j + 56^\circ = 180^\circ$$
$$146^\circ + j = 180^\circ$$

$$j = 180^\circ - 146^\circ$$
$$j = 34^\circ$$

$$h = 90^\circ \quad \text{radius} \perp \text{tangent}$$
$$k + j = 90^\circ \quad \text{radius} \perp \text{tangent}$$
$$k + 34^\circ = 90^\circ$$
$$k = 90^\circ - 34^\circ$$
$$k = 56^\circ$$

3. Find $\angle ABC$ in the given circle with centre O.



Solution:

Step 1. Assess and extract the given information from the problem. Given $\angle$ ADO = $36°$, DA $\perp$ OA; ΔOAD is a right-angled triangle

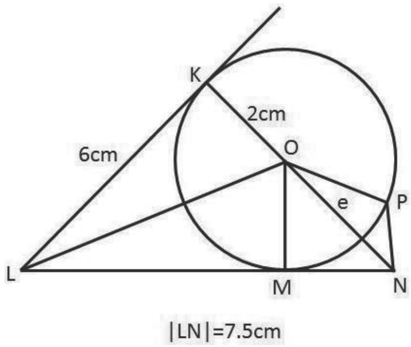
Step 2. Use theorems and the given information to find all equal angles on the diagram.

$\angle OAD$	$=$	$90°$	radius $\perp$ tangent
$\angle AOD + 36 + 90$	$=$	180	$\angle$s in a triangle
$\angle AOD$	$=$	$180 - 36 - 90$	
	$=$	54	

Step 3. Solve for $\angle$ ABC

$$\angle ABC = \frac{1}{2} \times 54° \quad \angle \text{ at the centre} = 2\angle \text{ at the circumference}$$
$$\angle ABC = 27°$$

4. Find the value of e in the diagram, where the circle has centre O.



Solution:

Given: $|LM| = 6$ cm, $|OK| = 2$ cm$|LN| = 7.5$ cm

$|LM| = |LK|$ equal tangents from same point

$|MN| = |LN| - |LM|$

$= 7.5 - 6$

$|MN| = 1.5$ cm

$|OM| = 2$ cm equal radii

From Δ OMN;

$e^2 = |OM|^2 + |MN|^2$ Pythagoras' theorem

$e^2 = 2^2 + 1.5^2$

$= 4 + 2.25$

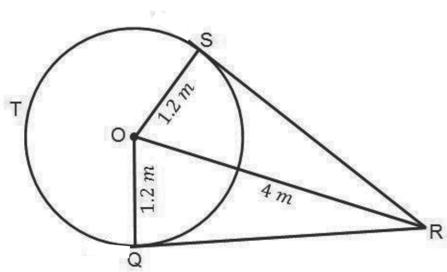
$e^2 = 6.25$

$e = \sqrt{6.25}$

$e = 2.5$ cm

5. The diagram below shows a belt QRST round a shaft R (of negligible radius) and a pulley of radius 1.2 m. O is the centre of the pulley; |OR|=4m and the straight portions QR and RS of the belt are tangents at Q and S to the pulley. Calculate:
   a. Angle QOS, correct to the nearest degree.
   b. The length of the belt(QRST) to the nearest metre. Use $\pi = 3.142$

Solution:



a.

Join O to R $= OR$

$\cos\angle SOR = \frac{1.2}{4}$ From ΔSOR

$\cos\angle SOR = 0.3$

$\angle SOR = \cos^{-1}(0.3)$

$= 72.54^0$

$\angle SOR = \angle QOR$ symmetry

$\angle QOS = 2 \times \angle SOR$

$= 2 \times 72.54^0$

$= 145.08^0$

$= 145^0$ to the nearest degree

b.

SR = QR symmetry

$OR^2 = SR^2 + OS^2$ Pythagoras' theorem

$4^2 = SR^2 + 1.2^2$

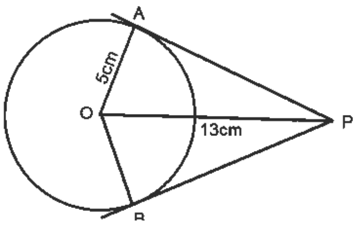
$$16 = SR^2 + 1.44$$
$$16 - 1.44 = SR^2$$
$$SR^2 = 14.56$$
$$SR = \sqrt{14.56}$$
$$= 3.82 \text{ m}$$

$$\text{Length of STQ} = \frac{360^0 - 145^0}{360^0} \times 2 \times 3.142 \times 1.2$$
$$= \frac{215^0}{360^0} \times 2 \times 3.142 \times 1.2$$
$$= \frac{1621.272}{360}$$
$$= 4.50 \text{ m}$$

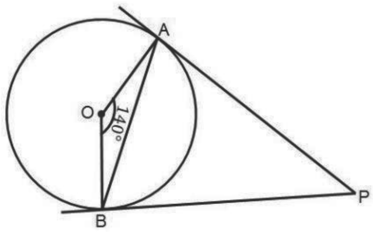
$$\text{Length of QRST} = \text{Length of QR + RS + STQ}$$
$$= 3.82 + 3.82 + 4.50$$
$$= 12.14$$
$$\text{Length of QRST} = 12 \text{ m to the nearest metre}$$

Practice

1. In the diagram, tangents PA and PB to a circle with centre O are drawn from a point P outside the circle. If the radius of the circle is 5 cm, and $|OP| = 13$ cm, find PA and PB.

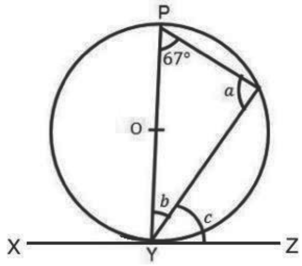


2. In the diagram, tangents PA and PB to a circle with centre O are drawn from a point P outside the circle. If $\angle AOB = 140°$, and AB is joined, find $\angle APB$ and $\angle ABP$.

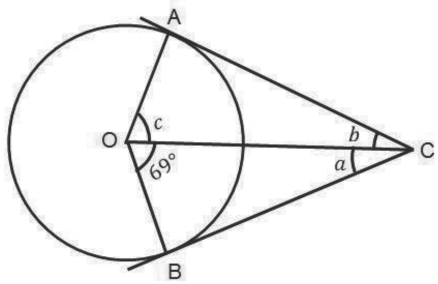


3. Find the angles marked with letters in the following diagrams. In each case, O is the centre of the circle.

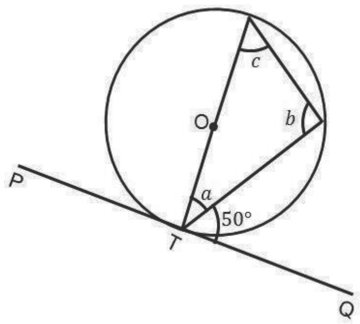
a.



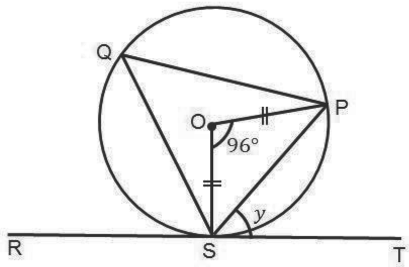
b.



c.



d.

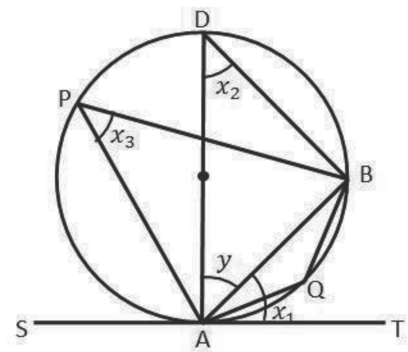


Lesson Title: Circle Theorem 8	Theme: Geometry
Practice Activity: PHM2-L071	**Class:** SSS 4

Learning Outcomes
By the end of the lesson, you will be able to:
1. Identify the alternate segment theorem.
2. Solve for missing angles using the alternate segment theorem.

Overview

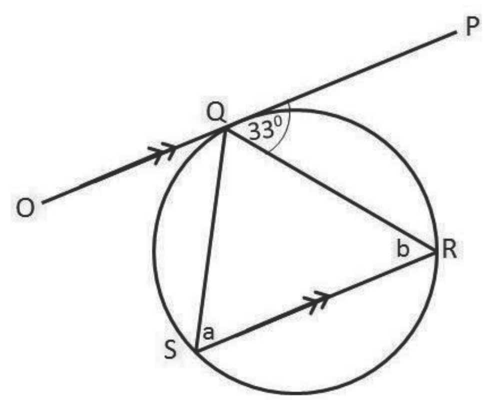
Circle Theorem 8: The angle between a chord and a tangent at the end of the chord equals the angle in the alternate segment (i.e. the angle in the other segment, not the one in which the first angle lies).



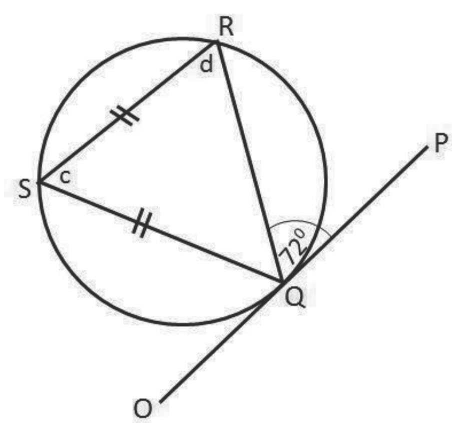
In the diagram, this means that:
- ∠TAB = ∠APB
- ∠SAB = ∠AQB

This is known as the alternate segment theorem.

Solved Examples

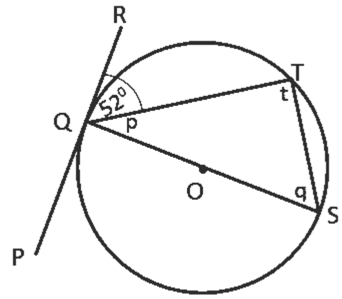


1. Find the missing angles a and b in the given circle at right.
   Solution:
   From circle theorem 8, angle b is equal to $\angle PQR$.
   $$a = \angle PQR = 33°$$
   Note that $\angle PQR$ and b are alternate angles because lines OP and SR are parallel. Therefore, angle b is also equal to $\angle PQR$.
   $$b = \angle PQR = 33°$$



2. Find the missing angles c and d in the given circle at right.
   Solution:
   From Circle Theorem 8, angle c is equal to $\angle PQR$.
   $$c = \angle PQR = 72°$$
   Note that ΔQRS is an isosceles triangle. Therefore, d is equal to ∠RQS. Calculate d:
   $$d = \frac{180-72}{2}$$
   $$d = 54°$$

3. Find the missing angle in the given circle at right with the centre O.



Solution:

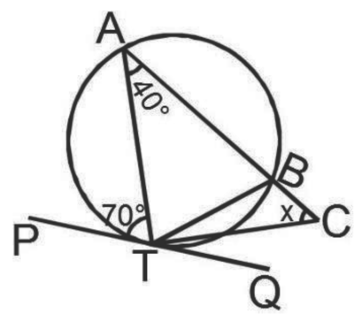
Each angle can be calculated using a circle theorem. If 2 angles are calculated, we can also apply subtraction from 180 to find the third angle of the triangle.

Using theorem 8, we have: $q = 52°$

∠QTS is an angle subscribed in a semi-circle, therefore $t = 90°$.

The tangent line PR is perpendicular to the radius, QO. Therefore, $p = 90 - 52 = 38°$.

4. In the diagram at right, PQ is a tangent to the circle at T. ABC is a straight line and TC bisects BTQ. Find x.

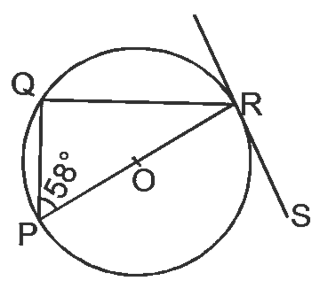


Solution:

∠ABT	=	$70°$	$\angle s$ in alternate segment
∠ABT+∠TBC	=	$180°$	Adjacent $\angle s$ on a straight line
70^0+∠TBC	=	$180°$	
∠TBC	=	$180 - 70$	
∠TBC	=	$110°$	
∠BTQ	=	40	$\angle s$ in alternate segment
∠BTC	=	$\frac{\angle BTQ}{2}$	TC bisects ∠BTQ

∠BCT+∠BTC+∠T	=	$180°$	sum of $\angle s$ in a Δ
x $+20° + 110°$	=	$180°$	
x $+130°$	=	$180°$	
x	=	$180° - 130°$	
x	=	$50°$	

5. In the diagram at right, PR is a diameter of the circle centre O. RS is a tangent at R and QPR = 58˚. Find ∠QRS.



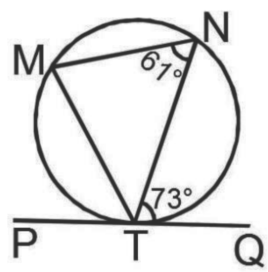
Solution:

∠PQR	=	$90°$	∠ in a semicircle
∠PRS	=	∠PQR	$\angle s$ in alternate segment
∠PRS	=	$90°$	
∠PQR+∠QPR+∠PRQ	=	180	sum of $\angle s$ in a Δ
$90 + 58 +$∠PRQ	=	180	

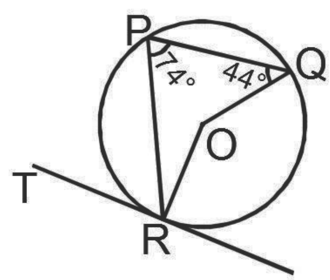
$$
\begin{aligned}
\angle \text{PRO}+148 &= 180 \\
\angle \text{PRQ} &= 180-148 \\
\angle \text{PRQ} &= 32 \\
\angle \text{QRS} &= \angle \text{PRQ}+\angle \text{PRS} \\
\angle \text{QRS} &= 32+90 \\
\angle \text{QRS} &= 122^\circ
\end{aligned}
$$

Practice

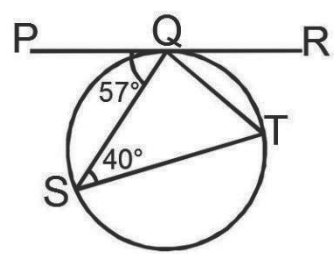
1. In the diagram at right, PQ is a tangent to the circle MTN at T. What is the measure of $\angle$MTN?



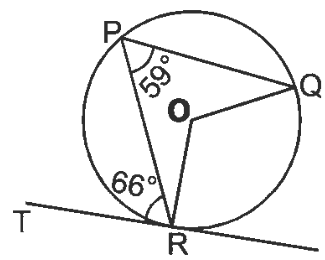
2. TR is the tangent to the circle PQR with centre O at right. Find the measure of $\angle$PRT.



3. In the diagram at right, PQR is a tangent to the circle SQT at Q. What is the measure of $\angle$SQT?



4. TR is the tangent to the circle PQR with centre O at right. Find the size of $\angle$PQO.



Lesson Title: Circle problem solving	**Theme:** Geometry
Practice Activity: PHM2-L072	**Class:** SSS 4

Learning Outcome
By the end of the lesson, you will be able to apply circle theorems and other properties to find missing angles in various circle diagrams.

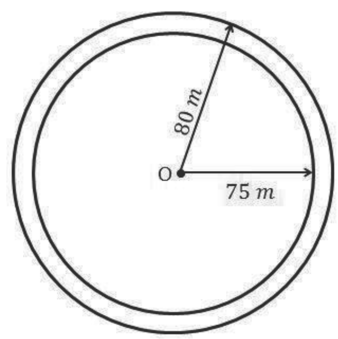
Overview

The problems in this lesson require you to apply theorems and other properties from lessons 67 through 71.

Solved Examples

1. There are five lanes in a circular running track. The radius of the edge of the track is 80 m; the radius of the first lane is 75 m. What is the difference in the distances run by two athletes if one runs around the edge of the track and the other runs around the first lane?

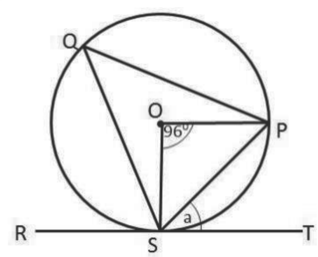
   Solution:



$$\begin{aligned} \text{Edge of track} &= 2 \times \tfrac{22}{7} \times 80 \text{ m} \\ &= \tfrac{3{,}520}{7} \\ &= 502.86 \text{ m}^2 \\ \text{First lane} &= 2 \times \tfrac{22}{7} \times 75 \text{ m} \\ &= \tfrac{3{,}300}{7} \\ &= 471.43 \text{ m}^2 \\ \text{Difference} &= \text{Distance of the edge of track} \\ &\quad -\text{Distance of first lane} \\ &= 502.86 \text{ m}^2 - 471.43 \text{ m}^2 \\ &= 31.43 \text{ m}^2 \end{aligned}$$

2. PQS is a circle with centre O. RST is a tangent at S and $\angle$SOP = 96°. Find $\angle$PST.

   Solution:
   The question can be solved in 2 ways using different circle theorems.



   Step 1. Assess and extract the given information from the problem.
   Given: RST is a tangent at S; $\angle$SOP = 96°
   Step 2. Use theorems and the given information to find all equal angles on the diagram.

Step 3. Write the answer.

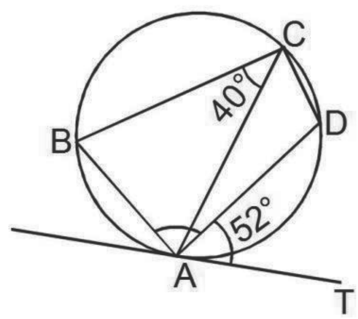
Method 1.

$$
\begin{aligned}
\angle PST &= \angle SQP && \angle s \text{ in alternate segment}\\
\angle SQP &= \tfrac{1}{2} \times \angle SOP && \angle \text{ at the centre} = 2\angle \text{ at circumference}\\
\angle SQP &= \frac{1}{2} \times 96 &&\\
&= 48° &&\\
\therefore \angle PST &= 48° &&
\end{aligned}
$$

Method 2.

$$
\begin{aligned}
OS &= OP && \text{equal radii}\\
2 \times \angle OSP + 96 &= 180 && \text{isosceles triangle}\\
\angle OSP &= \tfrac{180-96}{2} &&\\
&= 42° &&\\
\angle OST &= 90° && \text{radius} \perp \text{tangent}\\
\angle PST + 42 &= 90° &&\\
\angle PST &= 90 - 42 &&\\
PST &= 48° &&
\end{aligned}
$$

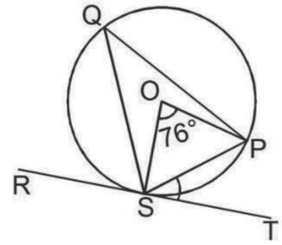
3. In the diagram, TA is a tangent to the circle at A. If $\angle BCA = 40^{\circ}$ and $\angle DAT = 52^{\circ}$, find $\angle BAD$.
   Solution:



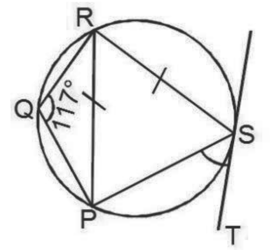
$$
\begin{aligned}
\angle ACD &= \angle DAT && \angle s \text{ in alternate segment}\\
\angle ACD &= 52^{\circ} &&\\
\angle BAD + \angle BCA + \angle ACD &= 180^{\circ} && \text{opposite } \angle s \text{ of a cyclic quadrilateral}\\
\angle BAD + \angle 40^0 + 52^0 &= 180^{\circ} &&\\
\angle BAD + \angle 92^0 &= 180^{\circ} &&\\
\angle BAD &= 180^{\circ} - 92^{\circ} &&\\
\angle BAD &= 88^{\circ} &&
\end{aligned}
$$

Practice

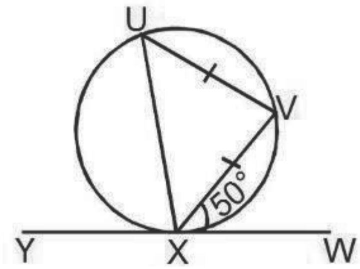
1. In the diagram, O is the centre of the circle. RT is a tangent at S and $\angle SOP = 76°$. Find $\angle PST$.



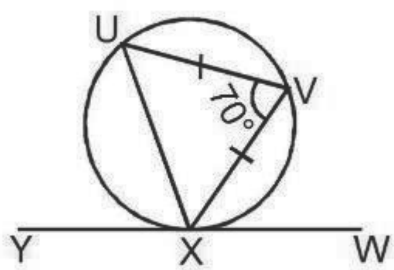
2. In the diagram, TS is a tangent to the circle at S, $|PR| = |RS|$ and ∠PQR = 117°. Calculate ∠PST.



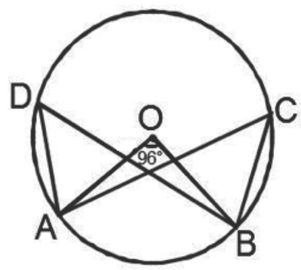
3. In the diagram, YW is a tangent to the circle at X, $|UV| = |VX|$ and ∠VXW = 50° Find the value of ∠UXY.



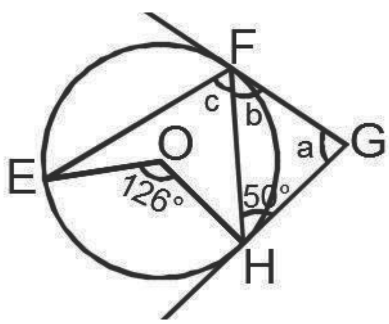
4. In the diagram, YW is a tangent to the circle at X, $|UV| = |VX|$ and ∠UVX = 70°. Find the value of ∠VXW.



5. In the figure below, ABCD are four points on the circumference of a circle, with centre O. If $\angle AOB = 96°$, find:
   a. $\angle ACB$
   b. $\angle BDA$



6. In the figure below, GF and GH are tangents to the circle with centre O, at points F and G respectively. If $\angle EOH = 126°$ and $\angle FHG = 50°$, find the angles marked $a, b,$ and c.



Lesson Title: Surface area	**Theme:** Mensuration
Practice Activity: PHM4-L073	**Class:** SSS 4

Learning Outcomes

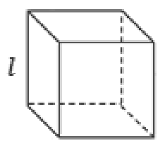
By the end of the lesson, you will be able to:

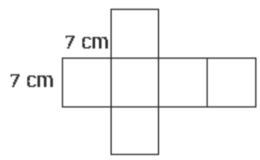
1. Identify the formulae for surface area.
2. Find the surface area of cubes, cuboids, prisms, cylinders, cones, pyramids, spheres and composite solids.

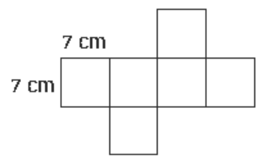
Overview

The surface area of a solid is the total area of its outside surface. The surface area is the **sum of the areas** of the faces of the three-dimensional shape.

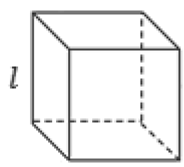
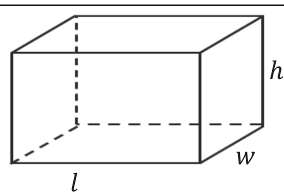
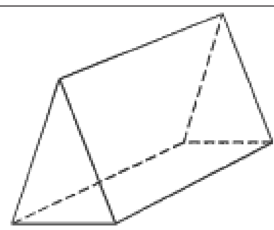
For example, a cube has 6 faces. The surface area is the sum of the areas of these 6 faces. We can visualise the surface area by drawing a net of the cube:

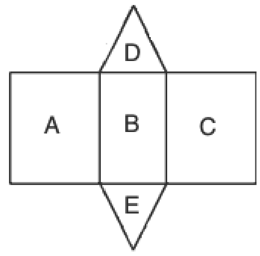
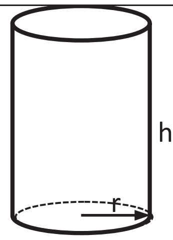
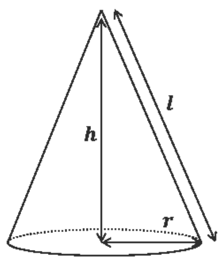
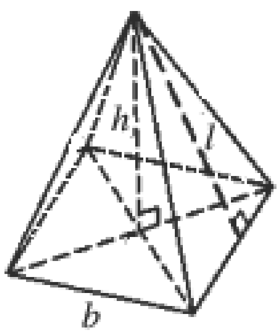
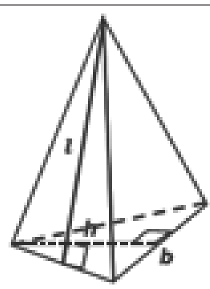


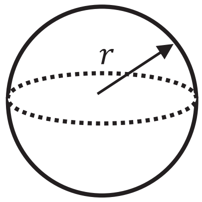




The basic method of finding surface areas of any solid is to find the area of the individual faces, then add up the areas. For some common solids, surface area can be calculated with formulae, which are listed:

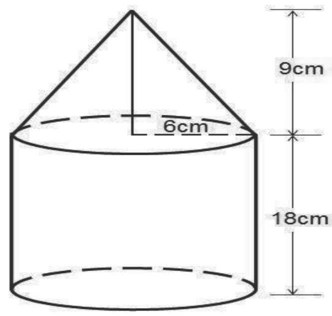
Solid	Diagram	Formula
Cube		$SA = 6 \times l^2$ where l is the length of the sides of the square faces.
Cuboid		$SA = lh + +lh + hw + hw + lw + lw$ $= 2(lh + hw + lw)$ where l is the length, w is the width, and h is the height.
Triangular Prism		There is no formula. Find the area of the individual faces, then add up the areas. For the prism with net:

		 SA =area of rectangle A + area of rectangle B + area of rectangle C + area of triangle D + area of triangle E
Cylinder		$SA = 2\pi r^2 + 2\pi rh$ $= 2\pi r(r + h)$ where r is the radius, and h is the height.
Cone		$SA = \pi rl + \pi r^2$ $= \pi r(l + r)$ where r is the radius, l is the length of the slanted surface, and h is the height.
Pyramid with a rectangular base		For most pyramids with a rectangular base, there is not a formula. You must find the area of each side and add them. A special case is the pyramid with a square base (pictured): $SA = b^2 + 2bl$ where b is the length of a side of the square, and l is the height of one of the slant edges.
Pyramid with a triangular base		$SA = \frac{1}{2}b(h + 3l)$ where b and h are the base and height of the triangular base, and l is the height of each of the other 3 triangular faces.

Sphere		$SA = 4\pi r^2$ where r is the radius.

To find the surface area of a composite solid, first identify the solids it is made up of. Find the individual surface area using the appropriate formula. Add the surface areas together. Be careful to account for faces that are covered, which do not count toward the surface area.

Solved Examples



1. The figure shown is a cylinder with a cone mounted on top. Calculate its total surface area correct to 1 decimal place. Use $\pi = \frac{22}{7}$.

   Solution:

   To find the total surface area of the figure, find the i) area of the curved surface of the cone and ii) area of curved surface of the cylinder and its base. Then add i) and ii).

   Total surface area of figure = area of curved surface of cone+area of curved surface of cylinder+area of base of cylinder

   area of curved surface of cone = $\pi r l$

   Use Pythagoras' theorem to find l:

   $$l^2 = 6^2 + 9^2$$
   $$l = 10.8 \text{ cm}^2$$

   area of curved surface of cone $= \frac{22}{7} \times 6 \times 10.8$

   $$= 203.7 \text{ cm}^2$$

   surface area of curved surface of cylinder+base $= 2\pi rh + \pi r^2$

   $$= 2 \times \frac{22}{7} \times 6 \times 18 + \frac{22}{7} \times 6^2$$
   $$= 792 \text{ cm}^2$$

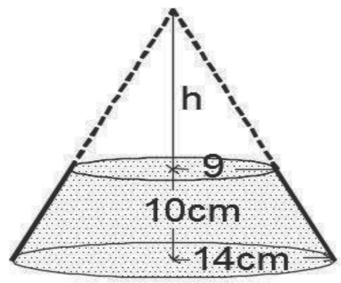
   Total surface area of the figure $= 203.7 \text{ cm}^2 + 792 \text{ cm}^2$

   $$= 995.7 \text{ cm}^2$$

2. A stage is to be made in the form of a cone with its top cut off. If the radius of the top of the stage is 9 cm and that of the base is 14 cm, calculate the total surface area of the top and sides of the stage. Use $\pi = \frac{22}{7}$.

Solution:

First, draw a picture. →



To find the total surface area of the stage, find i) the curved surface area of the frustum (the remaining part when the top of the full cone is removed), and ii) the area of the top of the frustum. Then add i) and ii).

The curved surface area of a cone= $\pi r l$, where r is the base radius, l is the slant height.

From the figure, let l_1 and l_2 be the respective slant heights of the cut off and complete cones. We can use similar triangles to find h and subsequently l_1 and l_2. From the figure, find h using similar triangles. Use the ratio of sides of the triangle:

$$\begin{aligned} \frac{h}{h+10} &= \frac{9}{14} \\ 14h &= 9h + 90 \\ 5h &= 90 \\ h &= 18\ cm \end{aligned}$$

Find l_1 and l_2 using Pythagoras' theorem:

$$\begin{aligned} l_1^2 &= h^2 + 9^2 \\ &= 18^2 + 9^2 \\ l_1 &= 20.1 \text{ cm} \end{aligned} \qquad \begin{aligned} l_2^2 &= (h+10)^2 + 14^2 \\ &= 28^2 + 14^2 \\ l_2 &= 31.3 \text{ cm} \end{aligned}$$

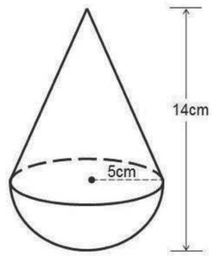
Note that the surface area of the stage is the curved surface area of the full cone, minus the curved surface of the small cone, plus the circular face of the small cone (the top of the stage). This gives: $SA = \pi r_2 l_2 - \pi r_1 l_1 + \pi r_1^2$, where r_2 and l_2 are measurements of the larger cone, and r_1 and l_1 are measurements of the smaller cone.

Substitute $\pi = \frac{22}{7}, r_2 = 14 \text{ cm}, r_1 = 9 \text{ cm}, l_{2=}31.3 \text{ cm},\ l_1 = 20.1 \text{ cm}$, and evaluate:

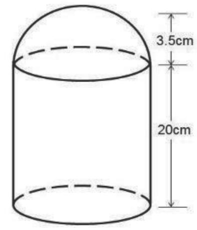
$$\begin{aligned} \text{Total surface area of stage} &= \pi r_2 l_2 - \pi r_1 l_1 + \pi r_1^2 \\ &= \left(\tfrac{22}{7}\right)(14)(31.3) - \left(\tfrac{22}{7}\right)(9)(20.1) + \left(\tfrac{22}{7}\right)(9)^2 \\ &= 1{,}377.2 - 568.5 + 254.6 \\ &= 1{,}063.3 \text{ cm}^2 \end{aligned}$$

Practice

1. Find the total surface area of a sphere of: a. radius 7 cm, b. diameter 3.5 m. Use $\pi = \frac{22}{7}$.
2. Find the total surface area of a cuboid with a length of 8 cm, a height of 5 cm, and a width of 2 cm, when a. It is open at the top; b. It is open at the top and bottom.
3. A solid is in the form of a cone attached to a hemisphere as shown in the figure at right. Find the total surface area of the figure.



4. The figure at right is a cylinder surmounted by a hemisphere. Find the total surface area of the figure.



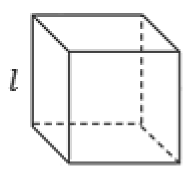
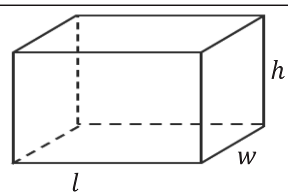
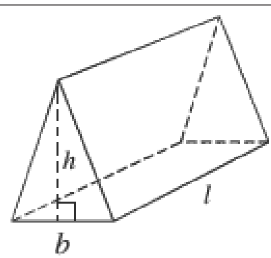
5. Find the total surface area of a square base pyramid where the square base has sides of length 12 cm, and the height of a slanting face is 20 cm. Give your answer correct to three significant figures.

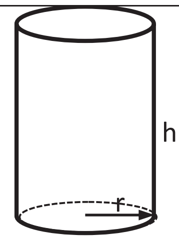
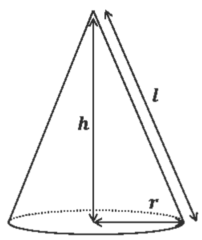
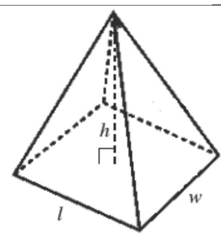
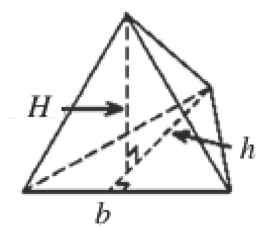
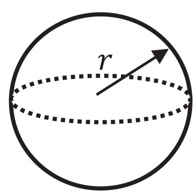
Lesson Title: Volume	**Theme:** Mensuration
Practice Activity: PHM4-L074	**Class:** SSS 4

Learning Outcomes
By the end of the lesson, you will be able to:
1. Identify the formulae for volume.
2. Find the volume of cubes, cuboids, prisms, cylinders, cones, pyramids, spheres and composite solids.

Overview

The volume of a three-dimensional solid is a measurement of the space occupied by the shape. This involves multiplication. The volume of common shapes can be found with formulae, which are given below.

Solid	Diagram	Formula
Cube		$V = l^3$ where l is the length of a side of the cube. If we know the area A of a face of the cube, then the volume is: $V = Al$
Cuboid		$V = lwh$ where l and w are the length and width of the base, and h is the height of the cuboid. If we know the area A of the base of the cuboid, then the volume is: $V = Ah$
Triangular Prism		$V = \frac{1}{2}bhl$ where b and h are the base and height of the triangular face, and l is the length of the prism. If we know the area A of the triangular face of the prism, then the volume is $V = Al$.

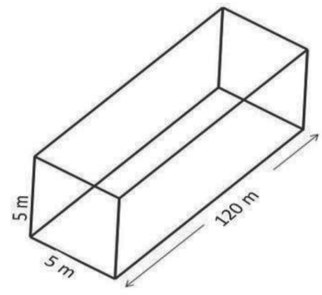
Cylinder		$V = \pi r^2 \times h$ where r is the radius of the circular face and h is the height of the cylinder. If we know the area A of the circular face of the cylinder then the volume is $V = Ah$, where h is the height.
Cone		$V = \frac{1}{3}\pi r^2 h$ where r is the radius and h the height of the cone.
Pyramid with a rectangular base		$V = \frac{1}{3} lwh$ where l and w are the length and width of the base rectangle, and h is the height of the pyramid.
Pyramid with a triangular base		$V = \frac{1}{3} AH$ where A is the area of the triangular base, and H is the height of the pyramid.
Sphere		$V = \frac{4}{3}\pi r^3$ where r is the radius.

To find the volume of a composite solid, first identify the solids it is made up of. Find the individual volume using the appropriate formula. Finally, add the volumes together.

For some WASSCE problems, you may need to use the fact that $1{,}000 \text{ cm}^3$ contains 1 litre of liquid ($1\ l = 1{,}000 \text{ cm}^3$).

Solved Examples

1. A tunnel with a length of 120 m is to be bored with a square cross-section 5 m wide and 5 m in height.
   a. What volume of spoil has to be evacuated?
   b. If the spoil is to be taken away using trucks of 80 m^3 capacity, how many truck loads will be moved?



Solutions:

The tunnel is in the form of a cuboid with a cross-sectional area of 5 m $\times$ 5 m and a length of 120 m. Draw a diagram. Note that the diagram shown is not proportional, but it is sufficient to visualise the problem.

a. Calculate the volume:

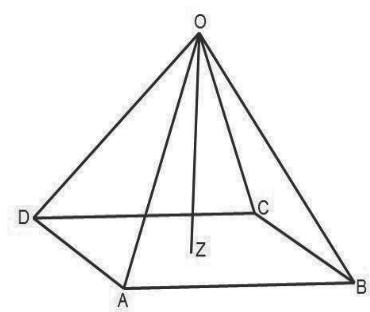
$$\begin{aligned} \text{Volume of spoil} &= \text{Area of cross-section} \times \text{length of tunnel} \\ \text{Area of cross-section} &= 5 \text{ m} \times 5 \text{ m} \\ &= 25 \text{ m}^2 \\ \text{Volume of spoil} &= 25 \times 120 \text{ m}^3 \\ &= 3{,}000\text{m}^3 \end{aligned}$$

b. Calculate the truck loads: If the capacity of one truck is 80 m^3 and n trucks are used to clear the $3{,}000 \text{ m}^3$, then:

$$\begin{aligned} 80n &= 3{,}000 \\ n &= 37\frac{1}{2} \end{aligned}$$

$37\frac{1}{2}$ truck loads will be moved.

2. The diagram shows a right pyramid with rectangular base ABCD and vertex O. If |AB|$=10$ cm, |DA|$=8$ cm and |OD|$=12$ cm, Find the:
   a. Height of the pyramid.
   b. Measure of $\angle ODB$.
   c. Volume of the pyramid.



Solutions:

At times, calculating volume is only 1 step of a given problem. We must use other Maths topics, including Pythagoras' theorem and trigonometry, to solve the other steps.

a. The height of the pyramid OZ is obtained from ΔOZD, which is right angled at Z. We must find the length of DZ, which is half of the diagonal of the base.
   DZ $= \frac{1}{2}$(diagonal of the base).
   Calculate DB, the length of the full diagonal, using triangle DAB:

$$|DB| = \sqrt{|AB|^2 + |DA|^2}$$
$$= \sqrt{10^2 + 8^2}$$
$$= \sqrt{164} = 2\sqrt{41}$$

Divide the result by 2 to find the length of DZ: $|DZ| = 2\sqrt{41} \div 2 = \sqrt{41}$

Calculate the height using triangle OZD:

$$|OZ|^2 + |DZ|^2 = |OD|^2 \quad \text{Using Pythagoras' theorem on } \Delta OZD$$
$$|OZ| = \sqrt{|OD|^2 - |DZ|^2} \quad \text{Height of pyramid (from } \Delta ODZ)$$
$$= \sqrt{12^2 - \left(\sqrt{41}\right)^2}$$
$$= \sqrt{103}$$
$$\text{Height} = 10.15 \text{ cm}$$

b. Apply the sine ratio to ΔODZ:

$$\sin D = \frac{\text{OZ}}{\text{OD}}$$
$$= \frac{10.15}{12}$$
$$= 0.8458$$
$$D = 57.8° \quad \text{Using the sine table}$$

c. Calculate the volume:

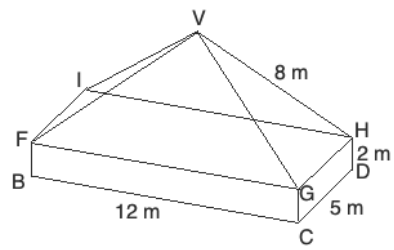
Volume of pyramid = $\frac{1}{3}$ ×(Area of base) × height

$$V = \frac{1}{3} \times |\text{DA}| \times |\text{AB}| \times |\text{OZ}|$$
$$V = \frac{1}{3} \times 8 \times 10 \times 10.15$$
$$= 270.67 \text{ cm}^3$$

Practice

1. Oil fills an inverted metal cone to a depth of 24 cm.
    a. If the radius of the surface of the oil is 20 cm, find the volume of oil. Assume that $\pi = 3.14$.
    b. The oil is then poured into a rectangular can of base 25 cm by 18 cm. Find, the depth of oil in the can correct to two decimal places.

2. A steel cuboid measuring 94.2 mm by 36 mm by 16 mm is melted down and cast into ball bearings of radius 6 mm. How many ball bearings are cast? $(\pi = 3.14)$
3. The diagram shows a pyramid standing on a cuboid. The dimensions of the cuboid are 2 m x 5 m x 12 m, and the slant edge of the pyramid is 8 m. Calculate the volume of the shape, correct to the nearest whole number.



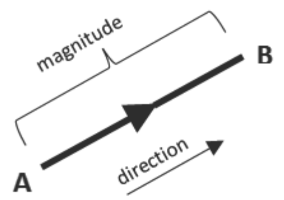
4. A right pyramid VABCD has a rectangular base ABCD and vertex V. |AB|= 10 cm, |AD|= 6 cm, and each of the slant sides is 8 cm long.
    a. Draw a diagram.
    b. Calculate:
        i. The height of the pyramid, correct to 2 decimal places.
        ii. The angle that VA makes with AC, correct to 1 decimal place.
        iii. The volume of the pyramid, correct to 2 decimal places.

Lesson Title: Operations on vectors	Theme: Vectors and Transformation
Practice Activity: PHM4-L075	**Class:** SSS 4

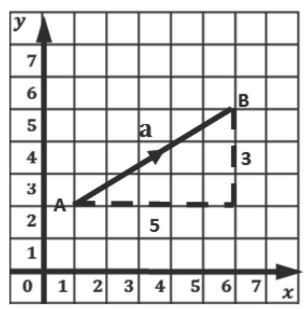
Learning Outcomes
By the end of the lesson, you will be able to:
1. Add and subtract vectors.
2. Multiply a vector by a scalar.

Overview

A **vector** is any quantity which has both magnitude and direction**.** Examples of vectors are displacement (translation), velocity, force. A **scalar** is any quantity which has only magnitude but no direction. Examples of scalars are distance, speed, time.
Vectors are represented in various ways. The simplest representation is as a line segment with a length equal to the magnitude of the vector and an arrow indicating its direction.



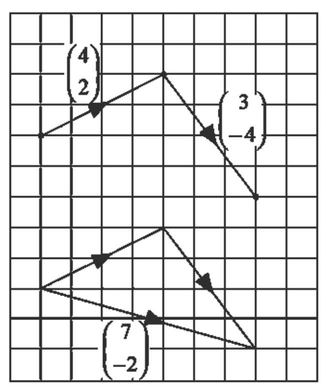
The vector on the right shows a displacement of a point from position A to position B. It can be written as: $\overrightarrow{AB}, \overline{AB}, \mathbf{AB}, \vec{\text{a}},\ \bar{\text{a}},\ \underset{\sim}{\text{a}},\ \mathbf{a}.$
Vectors written in lowercase letters are called position vectors.



Vectors can be represented on a Cartesian plane as shown at right. Consider the vector $\overrightarrow{AB}$: It can be written as a column matrix or column vector: $\overrightarrow{AB} = \mathbf{a} = \binom{5}{3}$**.** The vector is drawn by starting at point A, moving 5 units to the right and 3 units up.

In general, any vector $\overrightarrow{AB} = \binom{a}{b}$ has 2 components: the horizontal component a measured along the x-axis, and the vertical component, b measured along the y-axis from point A to point B.
Any move to the left or downwards is movement in the negative direction.

Addition and Subtraction of Vectors



Consider the vectors $\mathbf{a} = \binom{4}{2}$ and $\mathbf{b} = \binom{3}{-4}$ shown in the diagram on the right.

The result of adding **a** and **b** is as shown with the triangle below. To find the sum of 2 vectors, add the corresponding x and y components together:

$$\mathbf{a}+\mathbf{b} = \binom{4+3}{2+(-4)}$$
$$= \binom{7}{-2} \qquad \text{by calculation}$$

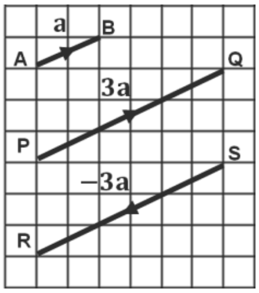
In general, if $\mathbf{a} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$, then

$$\mathbf{a} + \mathbf{b} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$$

Similarly,

$$\mathbf{a} - \mathbf{b} = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix}$$

Multiplication of a Vector and Scalar



Consider the vector $\overrightarrow{AB} = \mathbf{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ shown in the diagram,
It can be seen from the diagram that:

$\overrightarrow{PQ} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$
$= 3\begin{pmatrix} 2 \\ 1 \end{pmatrix}$
$= 3\mathbf{a}$ 3 times vector **a** in the same direction

$\overrightarrow{RS} = \begin{pmatrix} -6 \\ -3 \end{pmatrix}$
$= -3\begin{pmatrix} 2 \\ 1 \end{pmatrix}$
$= -3\mathbf{a}$ 3 times vector **a** in the opposite direction

In general, if $\mathbf{a} = \begin{pmatrix} x \\ y \end{pmatrix}$ then

$k\mathbf{a} = \begin{pmatrix} kx \\ ky \end{pmatrix}$ where k is a scalar or number which can be a positive or negative whole number or fraction

In scalar multiplication, each component of the vector is multiplied by the scalar amount. It has the effect of "scaling" the vector up or down by the factor of the scalar quantity. If the scalar is positive, the resulting vector is in the same direction as the original vector. If the scalar is negative, the resulting vector is in the opposite direction as the original vector.

Solved Examples

1. If **a**= $\begin{pmatrix} 4 \\ 7 \end{pmatrix}$, **b**= $\begin{pmatrix} 3 \\ -5 \end{pmatrix}$ and **c**= $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$, find:
   i. **a** + **b** ii. **b** + **c** iii. **a** – **c** iv. **a** + **b**– **c**

   Solution:

   i $\mathbf{a} + \mathbf{b} = \begin{pmatrix} 4 \\ 7 \end{pmatrix} + \begin{pmatrix} 3 \\ -5 \end{pmatrix}$
   $= \begin{pmatrix} 4+3 \\ 7+(-5) \end{pmatrix}$
   $= \begin{pmatrix} 7 \\ 2 \end{pmatrix}$

   ii. $\mathbf{b} + \mathbf{c} = \begin{pmatrix} 3 \\ -5 \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \end{pmatrix}$
   $= \begin{pmatrix} 3+0 \\ -5+4 \end{pmatrix}$
   $= \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

   iii. $\mathbf{a} - \mathbf{c} = \begin{pmatrix} 4 \\ 7 \end{pmatrix} - \begin{pmatrix} 0 \\ 4 \end{pmatrix}$
   $= \begin{pmatrix} 4-0 \\ 7-4 \end{pmatrix}$

   iv. $\mathbf{a} + \mathbf{b} - \mathbf{c} = \begin{pmatrix} 4 \\ 7 \end{pmatrix} + \begin{pmatrix} 3 \\ -5 \end{pmatrix} - \begin{pmatrix} 0 \\ 4 \end{pmatrix}$
   $= \begin{pmatrix} 4+3-0 \\ 7-5-4 \end{pmatrix}$

$$= \begin{pmatrix}4\\3\end{pmatrix} \qquad\qquad = \begin{pmatrix}7\\-2\end{pmatrix}$$

2. If **a**= $\begin{pmatrix}4\\2\end{pmatrix}$, **b**= $\begin{pmatrix}3\\-1\end{pmatrix}$., and **c**= $\begin{pmatrix}-2\\1\end{pmatrix}$, solve the equations below to find the column vector **x**.

   i. $\mathbf{a} + \mathbf{x} = \mathbf{b}$ ii. $\mathbf{x} - \mathbf{c} = \mathbf{a}$

   iii. $\mathbf{x} + \mathbf{b} = \mathbf{c}$ iv. $\mathbf{b} + \mathbf{x} = \mathbf{a}$

   Solution:

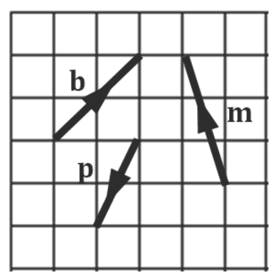
   i.
$$\begin{aligned} \mathbf{a} + \mathbf{x} &= \mathbf{b} \\ \begin{pmatrix}4\\2\end{pmatrix} + \mathbf{x} &= \begin{pmatrix}3\\-1\end{pmatrix} \\ \mathbf{x} &= \begin{pmatrix}3\\-1\end{pmatrix} - \begin{pmatrix}4\\2\end{pmatrix} \\ &= \begin{pmatrix}3-4\\-1-2\end{pmatrix} \\ &= \begin{pmatrix}-1\\-3\end{pmatrix} \end{aligned}$$

   ii.
$$\begin{aligned} \mathbf{x} - \mathbf{c} &= \mathbf{a} \\ \mathbf{x} - \begin{pmatrix}-2\\1\end{pmatrix} &= \begin{pmatrix}4\\2\end{pmatrix} \\ \mathbf{x} &= \begin{pmatrix}4\\2\end{pmatrix} + \begin{pmatrix}-2\\1\end{pmatrix} \\ &= \begin{pmatrix}4+(-2)\\2+1\end{pmatrix} \\ &= \begin{pmatrix}2\\3\end{pmatrix} \end{aligned}$$

   iii.
$$\begin{aligned} \mathbf{x} + \mathbf{b} &= \mathbf{c} \\ \mathbf{x} + \begin{pmatrix}3\\-1\end{pmatrix} &= \begin{pmatrix}-2\\1\end{pmatrix} \\ \mathbf{x} &= \begin{pmatrix}-2\\1\end{pmatrix} - \begin{pmatrix}3\\-1\end{pmatrix} \\ &= \begin{pmatrix}-2-3\\1-(-1)\end{pmatrix} \\ &= \begin{pmatrix}-5\\2\end{pmatrix} \end{aligned}$$

   iv.
$$\begin{aligned} \mathbf{b} + \mathbf{x} &= \mathbf{a} \\ \begin{pmatrix}3\\-1\end{pmatrix} + \mathbf{x} &= \begin{pmatrix}4\\2\end{pmatrix} \\ \mathbf{x} &= \begin{pmatrix}4\\2\end{pmatrix} - \begin{pmatrix}3\\-1\end{pmatrix} \\ &= \begin{pmatrix}4-3\\2-(-1)\end{pmatrix} \\ &= \begin{pmatrix}1\\3\end{pmatrix} \end{aligned}$$

3. Using the vectors **b**, **p** and **m** from the grid shown, draw the following vectors on a grid. Label each vector and show its direction with an arrow.

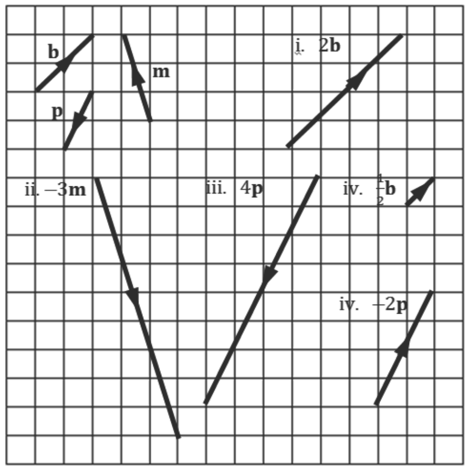


   i. 2**b** ii. –3**m** iii. 4**p**

   iv. $\frac{1}{2}$**b** v. –2**p**

   Solutions:

   Draw and label each vector on a grid showing its direction with an arrow. The position of the vectors relative to each other is not important. The length and direction of each vector should be as shown:



4. If **a**= $\binom{2}{1}$, **b**= $\binom{4}{-1}$ and **c**= $\binom{-2}{-4}$ find:

i. **3a** +2**b** ii. 4**a** + 3**c** iii. 6**a**– 3**b** iv. 5**a** +2**b**−4**c**

Solutions:

i.
$$\begin{aligned} 3\mathbf{a} + 2\mathbf{b} &= 3\binom{2}{1} + 2\binom{4}{-1} \\ &= \binom{6}{3} + \binom{8}{-2} \\ &= \binom{6+8}{3+(-2)} \\ &= \binom{14}{1} \end{aligned}$$

ii.
$$\begin{aligned} 4\mathbf{a} + 3\mathbf{c} &= 4\binom{2}{1} + 3\binom{-2}{-4} \\ &= \binom{8}{4} + \binom{-6}{-12} \\ &= \binom{8+(-6)}{4+(-12)} \\ &= \binom{2}{-8} \end{aligned}$$

iii.
$$\begin{aligned} 6\mathbf{a} - 3\mathbf{b} &= 6\binom{2}{1} - 3\binom{4}{-1} \\ &= \binom{12}{6} - \binom{12}{-3} \\ &= \binom{12-12}{6-(-3)} \\ &= \binom{0}{9} \end{aligned}$$

iv.
$$\begin{aligned} 5\mathbf{a} + 2\mathbf{b} - 4\mathbf{c} &= 5\binom{2}{1} + 2\binom{4}{-1} - 4\binom{-2}{-4} \\ &= \binom{10}{5} + \binom{8}{-2} + \binom{8}{16} \\ &= \binom{10+8+8}{5+(-2)+16} \\ &= \binom{26}{19} \end{aligned}$$

5. If **a**= $\binom{4}{2}$, **b**= $\binom{3}{-1}$, $\mathbf{c} = \binom{-2}{1}$ solve for $\mathbf{x}$ in the equations below.

i. 3**a**+2**x** =4**b**

ii. **4a**– **x**= **c**

Solutions:

i
$$\begin{aligned} 3\mathbf{a} + 2\mathbf{x} &= 4\mathbf{b} \\ 3\binom{4}{2} + 2\mathbf{x} &= 4\binom{3}{-1} \\ \binom{12}{6} + 2\mathbf{x} &= \binom{12}{-4} \\ 2\mathbf{x} &= \binom{12}{-4} - \binom{12}{6} \\ &= \binom{12-12}{-4-6} \\ &= \binom{0}{-10} \\ \mathbf{x} &= \binom{0}{-10} \div 2 \\ \mathbf{x} &= \binom{0}{-5} \end{aligned}$$

ii.
$$\begin{aligned} \mathbf{4a} - \mathbf{x} &= \mathbf{c} \\ 4\binom{4}{2} - \mathbf{x} &= \binom{-2}{1} \\ \binom{16}{8} - \mathbf{x} &= \binom{-2}{1} \\ \binom{16}{8} - \binom{-2}{1} &= \mathbf{x} \\ \binom{16-(-2)}{8-1} &= \mathbf{x} \\ \mathbf{x} &= \binom{18}{7} \end{aligned}$$

Practice

1. If **a**= $\begin{pmatrix}3\\5\end{pmatrix}$, **b**= $\begin{pmatrix}4\\-6\end{pmatrix}$, and **c**= $\begin{pmatrix}0\\7\end{pmatrix}$, find:
   i. $\mathbf{a}+\mathbf{b}$ ii. $\mathbf{b}+\mathbf{c}$ iii. $\mathbf{a}+\mathbf{b}-\mathbf{c}$
2. If **a**= $\begin{pmatrix}3\\2\end{pmatrix}$, **b**= $\begin{pmatrix}5\\-1\end{pmatrix}$, and **c**= $\begin{pmatrix}-4\\-6\end{pmatrix}$, find:
   i. $\mathbf{a}+\mathbf{b}$ ii. $\mathbf{a}+\mathbf{c}$ iii. $\mathbf{a}+\mathbf{c}+\mathbf{b}$ iv. $\mathbf{a}+\mathbf{b}-\mathbf{c}$
3. If **p**= $\begin{pmatrix}2\\4\end{pmatrix}$, **q**= $\begin{pmatrix}5\\1\end{pmatrix}$, and **r**= $\begin{pmatrix}9\\7\end{pmatrix}$, find:
   i. $\mathbf{p}+\mathbf{r}$ ii. $\mathbf{r}-\mathbf{p}$ iii. $\mathbf{r}-\mathbf{p}+\mathbf{q}$ iv. $\mathbf{p}-\mathbf{r}-\mathbf{q}$
4. If **a**= $\begin{pmatrix}6\\4\end{pmatrix}$, **b**= $\begin{pmatrix}5\\-3\end{pmatrix}$, and **c**= $\begin{pmatrix}-2\\1\end{pmatrix}$, solve the equations below to find the column vector **x**.
   i. $\mathbf{a}+\mathbf{x}=\mathbf{b}$ ii. $\mathbf{x}-\mathbf{c}=\mathbf{a}$ iii. $\mathbf{x}+\mathbf{b}=\mathbf{c}$
5. If **a**= $\begin{pmatrix}1\\2\end{pmatrix}$, **b**= $\begin{pmatrix}2\\-1\end{pmatrix}$ and **c**= $\begin{pmatrix}0\\-2\end{pmatrix}$ evaluate:
   i. $\mathbf{a}+\mathbf{b}$ ii. $\mathbf{b}-\mathbf{c}$ iii. $\mathbf{a}+\mathbf{b}+\mathbf{c}$ iv. $\mathbf{a}-2\mathbf{b}$ v. $3\mathbf{a}-\mathbf{b}-2\mathbf{c}$
6. If **a**= $\begin{pmatrix}2\\1\end{pmatrix}$, **b**= $\begin{pmatrix}6\\-2\end{pmatrix}$ and **c**= $\begin{pmatrix}-3\\-5\end{pmatrix}$ find:
   i. $\mathbf{a}+2\mathbf{b}$ ii. $4\mathbf{a}+3\mathbf{c}$ iii. $6\mathbf{a}-3\mathbf{b}$ iv. $5\mathbf{a}+2\mathbf{b}-4\mathbf{c}$
7. If **a**= $\begin{pmatrix}6\\2\end{pmatrix}$, **b**= $\begin{pmatrix}5\\-1\end{pmatrix}$, $\mathbf{c}=\begin{pmatrix}-2\\1\end{pmatrix}$ solve for **x** in the equations below.
   i. $3\mathbf{a}+2\mathbf{x}=4\mathbf{b}$ ii. $3\mathbf{a}-2\mathbf{x}=2\mathbf{c}$ iii. $2\mathbf{x}+6\mathbf{b}=2\mathbf{c}$

Lesson Title: Magnitude and direction of vectors	**Theme:** Vectors and Transformation
Practice Activity: PHM4-L076	**Class:** SSS 4

Learning Outcomes

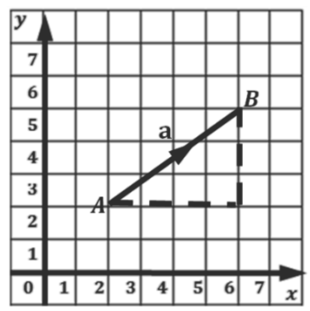
By the end of the lesson, you will be able to:

1. Find the magnitude or length of a column vector.
2. Find the direction of a vector.

Overview

Calculating Magnitude

We can use Pythagoras' Theorem to find the magnitude or length of the vector $\overrightarrow{AB}$.



Consider the diagram showing the 2 points $A(2,2)$and $B(6,5)$.
From the diagram, we can write the column vector for AB as:
$\overrightarrow{AB} = \binom{6-2}{5-2} = \binom{4}{3}$

The magnitude of vector $\overrightarrow{AB}$ can be written with the modulus or absolute value notation. For example: $|\overrightarrow{AB}|$, $|AB||\mathbf{a}|$, $|\underline{a}|$.
Now, we already know that for any 2 points:

$A(x_1, y_1)$ and $B(x_2, y_2)$, vector $\overrightarrow{AB} = \binom{x_2-x_1}{y_2-y_1} = \binom{x}{y}$.

So, since $\overrightarrow{AB} = \mathbf{a} = \binom{x}{y}$ then:

$$|\overrightarrow{AB}|^2 \quad = \quad |\mathbf{a}| \quad = \quad \sqrt{x^2 + y^2} \qquad (1)$$

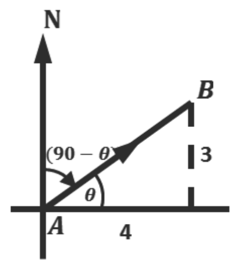
Alternatively, we can find the magnitude of $\overrightarrow{AB}$ by substituting directly in equation (1) using the co-ordinates of the given points:

$$|\overrightarrow{AB}|^2 \quad = \quad |\mathbf{a}| \quad = \quad \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \qquad (2)$$

The magnitude of $\overrightarrow{AB}$ in the diagram is:

$$\begin{aligned} |\overrightarrow{AB}|^2 &= \sqrt{x^2 + y^2} && \text{Pythagoras' Theorem} \\ &= \sqrt{4^2 + 3^2} \\ &= \sqrt{16 + 9} \\ |\overrightarrow{AB}| &= \sqrt{25} \\ &= 5 \ \text{ units} \end{aligned}$$

Calculating direction



The direction of the vector is given by the angle it makes when measured from the north in a clockwise direction. We find this angle by first finding the acute angle θ, the vector makes with the x-axis. It helps to first draw a sketch, as shown at right.
This angle is given by $\tan\theta = \frac{y}{x}$, where x, y are the components of the resultant vector. From our sketch, we can then deduce the angle the vector makes when measured from the north in a clockwise direction. In our example, this angle is given by $(90 - \theta)$. This is the same as finding the bearing of B from A.

Find the direction (bearing) of $\overrightarrow{AB}$ as follows:

Step 1. Find the measure of θ:

$$\begin{aligned} \tan\theta &= \frac{3}{4} = \quad 0.75 \qquad \text{from diagram, use tan ratio} \\ \theta &= \tan^{-1}(0.75) \\ &= 36.87° \end{aligned}$$

Step 2. Find the direction of $\overrightarrow{AB}$ measured from the north:

$$\begin{aligned} &= 90 - 36.87 \\ &= 53.13° \end{aligned}$$

The direction of $\overrightarrow{AB}$ measured from the north $= 53°$ to the nearest degree.

Solved Examples

1. Find the magnitude of the vectors: i. $\overrightarrow{AB} = \binom{3}{4}$ ii. $\overrightarrow{BC} = \binom{5}{0}$ iii. $\overrightarrow{CD} = \binom{-3}{4}$

Solutions:

a.
$$\begin{aligned} |\overrightarrow{AB}| &= \sqrt{x^2 + y^2} \\ &= \sqrt{3^2 + 4^2} \\ &= \sqrt{9 + 16} \\ &= \sqrt{25} \\ |\overrightarrow{AB}| &= 5 \end{aligned}$$

The magnitude of $\overrightarrow{AB} = 5$ units

b.
$$\begin{aligned} |\overrightarrow{BC}| &= \sqrt{x^2 + y^2} \\ &= \sqrt{5^2 + 0^2} \\ &= \sqrt{25 + 0} \\ &= \sqrt{25} \\ |\overrightarrow{BC}| &= 5 \end{aligned}$$

The magnitude of $\overrightarrow{BC} = 5$ units

c.
$$\begin{aligned} |\overrightarrow{CD}| &= \sqrt{x^2 + y^2} \\ &= \sqrt{-3^2 + 4^2} \\ &= \sqrt{9 + 16} \\ &= \sqrt{25} \\ |\overrightarrow{CD}| &= 5 \end{aligned}$$

The magnitude of $\overrightarrow{CD} = 5$ units

2. A column vector$\binom{x}{6}$ has a magnitude of 10. Find x.

   Solution:

   Given: magnitude of$\binom{x}{6} = 10$

$$\begin{aligned} \sqrt{x^2 + y^2} &= 10 \\ \sqrt{x^2 + 6^2} &= 10 \end{aligned}$$

   Square both sides

$$\begin{aligned} x^2 + 36 &= 100 \\ x^2 &= 100 - 36 \\ x^2 &= 64 \\ x &= \sqrt{64} \\ &= 8 \end{aligned}$$

3. If $\overrightarrow{XY} = \binom{2}{1}$ and $\overrightarrow{ZY} = \binom{3}{-5}$, find:

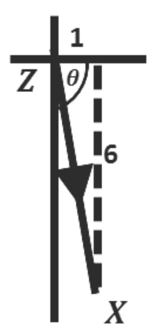
   a. $\overrightarrow{ZX}$

   b. The three-point bearing of X from Z correct to the nearest degree

   Solutions:

   a.
$$\begin{aligned} \overrightarrow{YZ} &= -\binom{3}{-5} \\ &= \binom{-3}{5} \\ \overrightarrow{XZ} &= \overrightarrow{XY} + \overrightarrow{YZ} \\ &= \binom{2}{1} + \binom{-3}{5} \\ &= \binom{2+(-3)}{1+5} \\ \overrightarrow{XZ} &= \binom{-1}{6} \\ \overrightarrow{ZX} &= -\binom{-1}{6} \\ &= \binom{1}{-6} \end{aligned}$$

   b. Find the acute angle θ, the vector makes with the x-axis



$$\begin{aligned} \tan\theta &= \frac{6}{1} \\ \theta &= \tan^{-1}(6) \\ &= 80.54° \end{aligned}$$

   The direction of $\overrightarrow{ZX}$ measured from the north:

$$\begin{aligned} &= 90 + 80.54 \\ &= 170.54° \end{aligned}$$

The direction of $\overrightarrow{ZX}$ measured from the north $= 171°$ to the nearest degree.
The bearing is $171°$.

4. The points $A(0,-1), B(4,-1), C(7,-2)$ and $D(3,-2)$ are the vertices of a parallelogram. Find:
   a. $\overrightarrow{AC}$
   b. The three-point bearing of C from A correct to the nearest degree

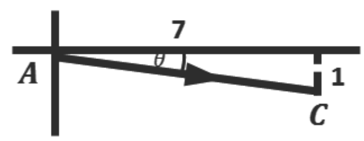
Solutions:

Let position vectors $\overrightarrow{OA} = \binom{0}{-1}$ and $\overrightarrow{OC} = \binom{7}{-2}$

a.
$$\begin{aligned}\overrightarrow{OA} + \overrightarrow{AC} &= \overrightarrow{OC} \\ \overrightarrow{AC} &= \overrightarrow{OC} - \overrightarrow{OA} \\ \overrightarrow{AC} &= \binom{7}{-2} - \binom{0}{-1} \\ &= \binom{7-0}{-2-(-1)} \\ \overrightarrow{AC} &= \binom{7}{-1}\end{aligned}$$

b. Find the acute angle θ, the vector makes with the x-axis

$$\begin{aligned}\tan\theta &= \frac{1}{7} \\ \theta &= \tan^{-1}\left(\frac{1}{7}\right) \\ &= 8.13°\end{aligned}$$



The direction of $\overrightarrow{AC}$ measured from the north:

$$\begin{aligned} &= 90 + 8.13 \\ &= 98.13°\end{aligned}$$

The direction of $\overrightarrow{AC}$ measured from the north $= 98°$ to the nearest degree.
The three-point bearing of C from A is $098°$.

Practice

1. Find the magnitude of the given vectors to 1 decimal place:
   a. $\overrightarrow{AB} = \binom{-5}{7}$ b. $\overrightarrow{RT} = \binom{2}{4}$
2. Find the magnitude of the vector $\mathbf{r} = \binom{-4}{12}$. Give your answer: a. in surd form; b. to 2 decimal places.
3. XYZ is a triangle with vertices X(1,-3),Y(7,5) and Z(-3,5).
   a. If O is the origin, express $\overrightarrow{XY}$, $\overrightarrow{YZ}$ and $\overrightarrow{ZX}$ as column vectors.
   b. show that triangle XYZ is isosceles.
4. A column vector$\binom{7}{y}$ has a magnitude of 25. Find y.
5. A column vector$\binom{x}{12}$ has a magnitude of 13. Find x.

6. Find the direction of the given vectors to the nearest whole number: a. $\overrightarrow{XY} = \binom{9}{-3}$; b. $\overrightarrow{PR} = \binom{-4}{-8}$
7. $P(-1,2)$ and $Q(x, y)$are points on the xy-plane such that $\overrightarrow{PQ} = \binom{3}{-4}$. Find:
    a. the coordinates of Q
    b. the bearing of P from Q to the nearest degree.
8. $A(1,2), B(4,6), C(2,7)$ and $D(x, y)$ are the vertices of the parallelogram ABCD. Find:
   a) x and y b) $\overrightarrow{AC}$ c) $\overrightarrow{CA}$
9. If $\overrightarrow{AB} = \binom{3}{-5}$ and $\overrightarrow{CB} = \binom{2}{1}$, find:
    a. $\overrightarrow{CA}$
    b. the bearing of C from A correct to the nearest degree.

Lesson Title: Transformation	**Theme:** Vectors and Transformation
Practice Activity: PHM4-L077	**Class:** SSS 4

Learning Outcome
By the end of the lesson, you will be able to perform transformations (reflection, rotation, translation, and enlargement).

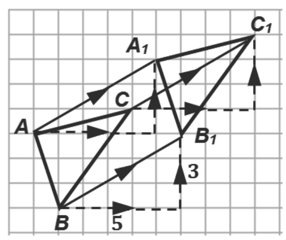
Overview

Transformation changes the position, shape, or size of an object. The following transformations are covered in this lesson:

- **Translation** – moves all the points of an object in the same direction and the same distance without changing its shape or size.
- **Reflection** – an object is reflected in a line of symmetry. The direction that it faces changes, but not its size.
- **Rotation** – an object rotates (or turns) around a point, which is called the centre of rotation.
- **Enlargement –** the object is magnified (made larger) or diminished (made smaller). Its shape does not change, but its size does.

Vectors are used to describe the transformation of objects. A combination of transformations can be applied to an object.

Translation



Consider the triangle *ABC* shown at right. It was translated from its original position 5 units right and 3 units up.
The column vectors which show the movements of *A, B* and *C* to their new positions A_1, B_1 and C_1 are given by: $\overrightarrow{AA_1} = \binom{5}{3}$, $\overrightarrow{BB_1} = \binom{5}{3}$ and $\overrightarrow{CC_1} = \binom{5}{3}$. The vector $\binom{5}{3}$ is called a **translation** vector, **v**.
In general a translation vector $\binom{a}{b}$ moves a point $A(x, y)$ along the x- and y-axes by the amount of the components of the vector. We can write a mapping for the translation as: $\binom{x}{y} \longrightarrow \binom{x}{y} + \binom{a}{b} = \binom{x+a}{y+b}$

Reflection

Every point on the reflected image is the same distance away from the line of reflection as the object. Distances from both object and image are always measured at right angles to the mirror line. The mappings for various reflections are below.

- **Reflection in the x-axis (i.e. $y = 0$)** is given by:
  $\binom{x}{y} \longrightarrow \binom{x}{-y}$ giving $(x, y) \longrightarrow (x, -y)$
- **Reflection in the y-axis (i.e. $x = 0$)** is given by:

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} -x \\ y \end{pmatrix} \quad \text{giving} \quad (x, y) \longrightarrow (-x, y)$$

- **Reflection in the line $y = k$ or $y - k = 0$** is given by:

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} x \\ 2k-y \end{pmatrix} \quad \text{giving} \quad (x, y) \longrightarrow (x, 2k - y)$$

- **Reflection in the line $x = k$ or $x - k = 0$** is given by:

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} 2k-x \\ y \end{pmatrix} \quad \text{giving} \quad (x, y) \longrightarrow (2k - x, y)$$

- **Reflection in the line** $y = x$ as:

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} y \\ x \end{pmatrix} \quad \text{giving} \quad (x, y) \longrightarrow (y, x)$$

- **Reflection in in the line** $y = -x$ as:

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} -y \\ -x \end{pmatrix} \quad \text{giving} \quad (x, y) \longrightarrow (-y, -x)$$

Rotation

Rotation is a movement around a fixed point, called the centre of rotation. Rotation is always done in a specified angle and direction about a specified point. The mappings for rotation are given below.

- **Rotation through $90°$ anti-clockwise or $270°$ clockwise about the origin O:**

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} -y \\ x \end{pmatrix} \quad \text{giving} \quad (x, y) \longrightarrow (-y, x)$$

- **Rotation through $270°$ anti-clockwise or $90°$ clockwise about the origin O:**

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} y \\ -x \end{pmatrix} \quad \text{giving} \quad (x, y) \longrightarrow (y, -x)$$

- **Rotation through $180°$ (half turn) anti-clockwise about the origin O:**

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} -x \\ -y \end{pmatrix} \quad \text{giving} \quad (x, y) \longrightarrow (-x, -y)$$

We follow the steps given below to find the formula for **rotation through $90°$ anti-clockwise or $270°$ clockwise about the point (a, b).**

Step 1.	Subtract the co-ordinates of the centre of rotation (a, b) from (x, y)	$\begin{pmatrix} x-a \\ y-b \end{pmatrix}$
Step 2.	Apply the appropriate rotation formula	$\begin{pmatrix} x-a \\ y-b \end{pmatrix} \longrightarrow \begin{pmatrix} -(y-b) \\ x-a \end{pmatrix}$
Step 3.	Add the result in Step 2 to the centre of rotation to get the image point	$\begin{pmatrix} -(y-b)+a \\ (x-a)+b \end{pmatrix}$
Step 4.	Write the co-ordinates of the image point	$(-(y - b) + a,\ (x - a) + b)$

Enlargement

An enlargement is a transformation which enlarges or reduces the size of an image. It is described by a centre of enlargement and a scale factor, k.Two different formulas are given for enlargement:

- The formula **for enlargement from the origin O by a scale factor k** is given by:

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow k\begin{pmatrix} x \\ x \end{pmatrix} = \begin{pmatrix} kx \\ kx \end{pmatrix}$$ where k is positive or negative whole number or fraction

- The formula for **enlargement from any point (a, b) other than the origin O by a scale factor k** can be found by following the steps given below.

Step 1. Subtract the co-ordinates of the centre of rotation (a, b) from (x, y) $\quad \binom{x-a}{y-b}$

Step 2. Enlarge using the given scale factor $\quad \binom{x-a}{y-b} \longrightarrow \binom{k(x-a)}{k(y-b)}$

Step 3. Add the result in Step 2 to the centre of rotation to get the image point $\quad \binom{k(x-a)+a}{k(y-b)+b}$

Step 4. Write the co-ordinates of the image point $\quad (k(x-a)+a,\ k(y-b)+b)$

A negative scale factor gives an inverted image at the opposite side of the centre of enlargement. A fractional scale factor gives a reduction, or smaller image.

Solved Examples

1. Triangle PQR has coordinates $P(1,4), Q(2,1)$ and $R(4,2)$. Find the co-ordinates, P_1, Q_1 and R_1 of the image of the triangle formed under reflection in the line $y = -x$.

Solution:

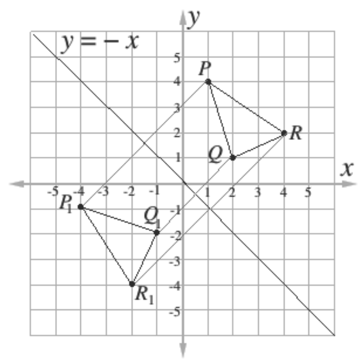
Step 1. Locate the points P, Q and R on the plane. Draw the lines joining the points.

Step 2. Draw the line $y = -x$

Step 3. Draw a line at right angles from P to the mirror line ($y = -x$). Measure this distance.

Step 4. Measure the same distance on the opposite side of the mirror line ($y = -x$) to locate the point P_1.

Step 5. Identify and write the new co-ordinates: $P_1(-4, -1)$, $Q_1(-1, -2)$ and $R_1(-2, -4)$



Note: Images can also be found using the formula

2. Use the appropriate formula to find the co-ordinates of the image point when point $X(-3, -2)$ is rotated $90°$ clockwise about the point $(0, -4)$.

Solution:

Apply the following formula for rotation:

$$\binom{x-a}{y-b} \longrightarrow \binom{(y-b)+a}{-(x-a)+b} = \binom{(y-(-4))+0}{-(x-0)+(-4)} = \binom{y+4}{-x-4}$$

$$\binom{-3}{-2} \longrightarrow \binom{-2+4}{-(-3)-4} = \binom{2}{-1} \qquad \text{apply the formula}$$

$X(-3, -2)$ rotated $90°$ clockwise about the point $(0, -4)$ gives $(2, -1)$

3. Points $A(2,4), B(6,3)$ and $C(3,1)$ are points on the Cartesian plane. Find the co-ordinates, A_1, B_1 and C_1 of the image of the triangle formed under an anti-clockwise rotation of $90°$ about the origin, O.

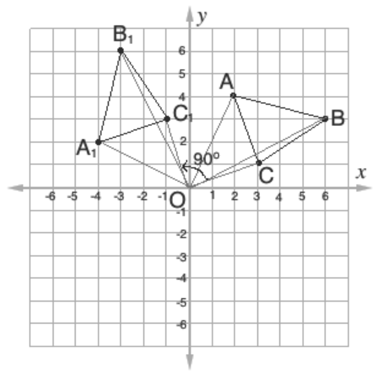
Solution:

Step 1. Draw the Cartesian plane and locate points $A,\ B$ and C. Draw lines connecting the points.

Step 2. Mark the centre of rotation $O(0,0)$. Draw a straight line from O to each point A, B, and C.

Step 3. Measure an angle of $90°$ in an anti-clockwise direction from each line you drew.

Step 4. Identify A_1, B_1 and C_1 on the plane: $A_1 = (-4,2)$, $B_1(-3,6)$ and $C_1(-1,3)$.



Note: Images can also be found using the formula

4. Find the image of $(1,-3)$ under the enlargement with scale factor of 3 from:
   a. The origin b. The point $(2,4)$

Solutions:

a. Multiply by the scale factor:

$$\binom{1}{-3} \longrightarrow 3\binom{1}{-3} = \binom{3\times(1)}{3\times(-3)} = \binom{3}{-9}$$

b. Apply the formula for enlargement:

$\binom{x-a}{y-b} \longrightarrow \binom{1-2}{-3-4} = \binom{-1}{-7}$ subtract components of the centre of rotation from the given point

$\binom{-1}{-7} \longrightarrow 3\binom{-1}{-7} = \binom{-3}{-21}$ enlarge using the given scale factor

$\binom{-3}{-21} \longrightarrow \binom{-3+2}{-21+4} = \binom{-1}{-17}$ add back components of the centre of rotation

5. Triangle ABC has co-ordinates $A(-2,4)$, $B(1,5)$ and $C(-1,1)$.
   a. Draw triangle ABC on the Cartesian plane.
   b. Draw triangle $A_1B_1C_1$, which is ABC translated by the vector $\binom{3}{-4}$.

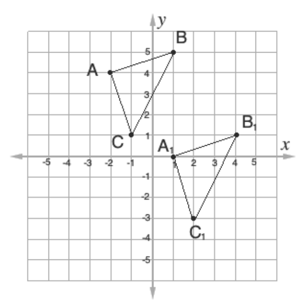
   Solutions:

   a. Identify on the Cartesian plane, and connect them in a triangle as shown.
   b. Identify point A_1 by translating $A(-2,4)$ by the vector $\binom{3}{-4}$:

   $\mathrm{A}\binom{-2}{4} \longrightarrow \binom{-2}{4} + \binom{3}{-4} = \binom{-2+3}{4-4} = A_1\binom{1}{0}$

   Similarly, $\mathrm{B}\binom{1}{5} = \mathrm{B}_1\binom{4}{1}$ and, $\mathrm{C}\binom{-1}{1} = C_1\binom{2}{-3}$

   Draw the image of $A_1B_1C_1$ using A_1 as a reference point. Note that it has the same shape and size as ABC.



Practice

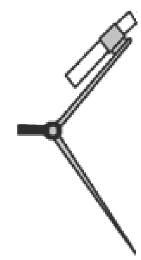
1. Triangle DEF has co-ordinates $D(-4,1), E(-2,3)$ and $F(-5,4)$. Find the co-ordinates, D_1, E_1 and F_1 of the image of the triangle formed under reflection in the line $y = x$.
2. Find the image of $(-5,7)$ under the enlargement with scale factor of 5 from the point $(-1,-1)$.
3. Triangle ABC has co-ordinates $A(-2,-4)$, $B(-2,-1)$ and $C(-5,-1)$. Draw triangle $A_1B_1C_1$, which is ABC translated by the vector $\binom{7}{5}$.
4. Points $A(2,4), B(6,3)$ and $C(3,1)$ are points on the Cartesian plane. Find the co-ordinates, A_1, B_1 and C_1 of the image of the triangle formed under a **clockwise** rotation of $90°$ about the origin, O.
5. A' $(-2,4)$ is the image of a point A under the translation by the vector $\binom{2}{-1}$. Find the co-ordinates of point A.
6. $P'(5\ ,2)$ is the image of the point $P(2\ ,-5)$ by the translation vector $\mathbf{v}$. Find:
    a. The vector $\mathbf{v}$.
    b. The co-ordinates of point Q which maps on to point $Q'(-5,-2)$ under $\mathbf{v}$.
7. A square has vertices $A(8,4)$, $B(8,-8)$, $C(-4,-8)$ and $D(-4,4)$. Find the co-ordinates of the vertices of the image square $A_1B_1C_1D_1$ of $ABCD$ under an enlargement from the origin with scale factor $\frac{1}{2}$ where $A \longrightarrow A_1$, $B \longrightarrow B_1$, $C \longrightarrow C_1$ and $D \longrightarrow D_1$.

Lesson Title: Bisection	**Theme:** Geometry
Practice Activity: PHM2-L078	**Class:** SSS 4

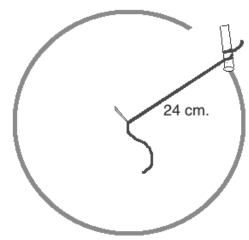
Learning Outcome
By the end of the lesson, you will be able to bisect a given line or angle.

Overview

This is the first lesson on geometry construction. The tool pictured at right is a compass. It makes circles, and can be used to construct many different angles and shapes in geometry. If you do not have a pair of compasses, you can make one using string or a strip of paper.

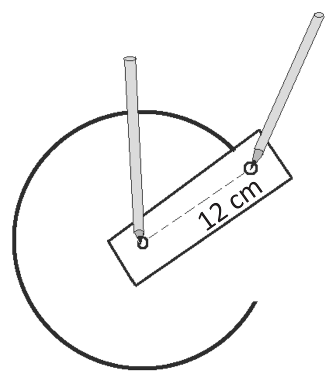


Using **string** to make a pair of compasses:



1. Cut a piece of string longer than the radius of the circle you will make.
2. Tie one end of the string to a pencil.
3. Hold the string to your paper. The distance between the place you hold and the pencil will be the **radius** of the circle. In the diagram, the radius of the circle is 24 cm.
4. Use one hand to hold the string to the same place on the paper. Use the other hand to move the pencil around and draw a circle.

Using **paper** to make a pair of compasses:



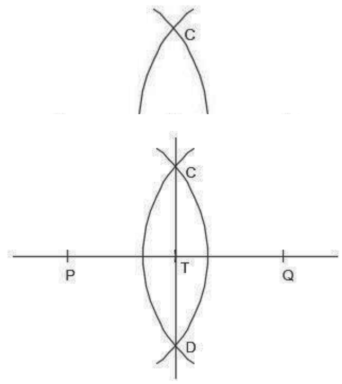
1. Cut or tear any piece of paper, longer than the radius of the circle you will make.
2. Make two small holes in the paper. The distance between the two holes will be the radius of your circle. In the diagram below, the radius of the circle is 12 cm.
3. Put something sharp (a pen or pencil will work) through one hole. Place this in your exercise book, at the centre of the circle you will draw.
4. Put your pen or pencil through the other hole, and move it in a circle on your paper.

This lesson uses a pair of compasses to construct bisectors of line segments and angles. To **bisect** something means to divide it into 2 equal parts. The bisection of a line segment is often called a “perpendicular bisector” because it is perpendicular to a given line, and divides it equally.

In the diagram below, $\overline{CD}$ is the perpendicular bisector of line segment $\overline{PQ}$. It is perpendicular to $\overline{PQ}$ at point T.

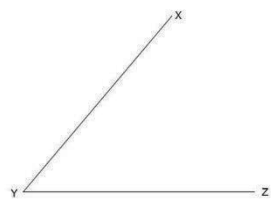
To construct this perpendicular bisector, follow these steps:

1. With point P as centre, open your compass more than half way to point Q. Then draw an arc above and below PQ.
2. Using the same radius and point Q as centre, draw an arc that intersects the first arc. Label the points where the 2 arcs intersect as C and D.
3. Draw $\overline{CD}$.



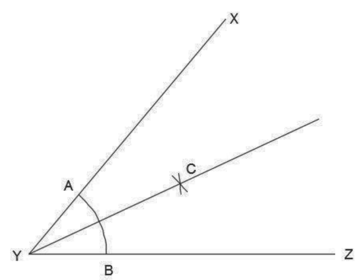
As with lines, we can bisect an angle by dividing it into 2 equal parts. For example, if an angle is 60°, we can bisect it to find an angle of 30°.

Consider angle XY:

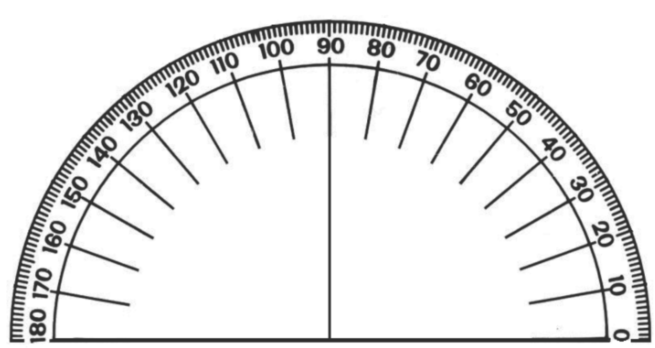


We can divide it into 2 equal angles. To bisect $\angle XYZ$, follow these steps:

1. With point Y as the centre, open your pair of compasses to any convenient radius. Draw an arc AB to cut XY at A and YZ at B.
2. With point A as centre, draw an arc using any convenient radius.
3. With the same radius as the step above, use point B as centre and draw another arc to intersect the first one at C.
4. Label point C.
5. Join Y to C to get the angle bisector as shown.



You can use a protractor to check the measure of the angles you draw. If you do not have a protractor, you can make one with paper. Trace this protractor with a pen onto another piece of paper.

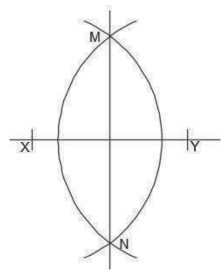


Let's check the bisection of $\angle XYZ$:

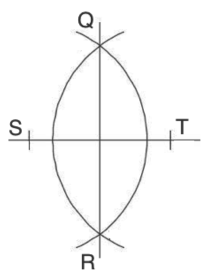
1. Hold the protractor to $\angle XYZ$, and measure the entire angle (it is $50°$).
2. Write down the angle measure.
3. Hold the protractor up again, and measure angles CYZ and XYC.
4. Write the measure of each bisection. (In this case, $\angle CYZ = 25°$ and $\angle XYC = 25°$)

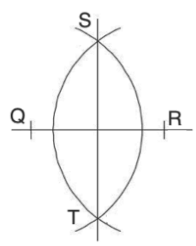
Solved Examples

1. Construct line XY with perpendicular bisector MN.
   Solution:



2. Line ST and QR are perpendicular lines. Construct the 2 lines.
   Solution:
   Either construction is correct:





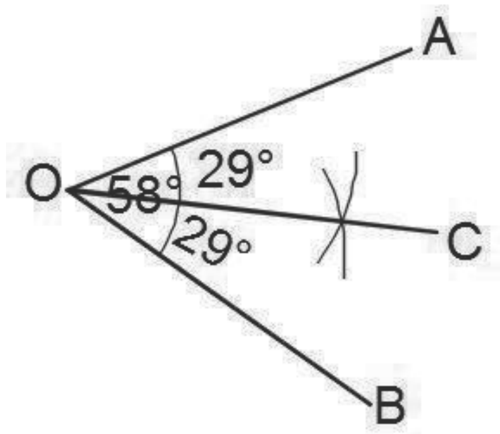
3. Line AB is 10 cm long. Line CD is its perpendicular bisector, and the 2 lines intersect at point O.
   a. What is the length of AO?
   b. What is the length of OB?
   Solution:
   This problem does not require a construction. We know that a perpendicular bisector divides a line into 2 equal segments. Therefore, $AO = OB$. Divide the length of AB by 2 to get the length of each.

      a. $|AO| = 10 \text{ cm } \div 2 = 5 \text{ cm}$
      b. $|OB| = 10 \text{ cm } \div 2 = 5 \text{ cm}$

4. Draw and label $< AOB = 58°$. On your angle:
   a. Construct a bisection, line OC.

b. Measure each bisection and label it with its degrees.

Solution:



$\angle\text{AOC} = 29°$ and $\angle\text{BOC} = 29°$

5. If you bisected a 45-degree angle, what would be the measure of the result?

   Solution:

   Bisecting an angle gives a result that is half of the original angle. Therefore, you would have $\frac{45°}{2} = 22.5°$

Practice

1. Construct line BO with perpendicular bisector SL.
2. Line MN is 24 metres long. If it is bisected at point O, what is:
   a. The length of MO?
   b. The length of ON?
3. Draw any angle and label it PQR. One your angle:
   a. Construct an angle bisector QU.
   b. Measure it bisector using a protractor and label each angle with its measurement.
4. Draw an angle labeled with three letters of your choice.
   a. Bisect the angle.
   b. Check your bisection using a protractor. Label each angle with its measurement.

Lesson Title: Angle construction	Theme: Geometry
Practice Activity: PHM4-L079	**Class:** SSS 4

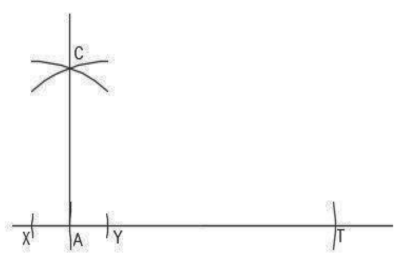
Learning Outcome
By the end of the lesson, you will be able to use a pair of compasses to construct special angles and their combinations (90°, 45°, 60°, 120°, 30°, 75°, 105°, and 150°).

Overview

This lesson is on constructing 8 angles of various degrees. These are all based on the construction of the angles 90°, 60°, and 120°. From these angles, you can construct all of the others using bisection.

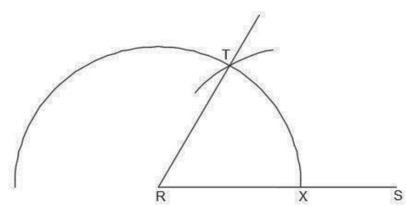
To construct a **90° angle**, follow these steps:

1. Draw a horizontal line and label it A.
2. Extend the straight line outwards from A.
3. With A as the centre, open your compass to a convenient radius and draw a semi-circle that intersects the line at X and Y.
4. Use X and Y as centres. Using any convenient radius, draw arcs to intersect at C.
5. Draw a line from A to C.



To construct a **60° angle**, follow these steps:

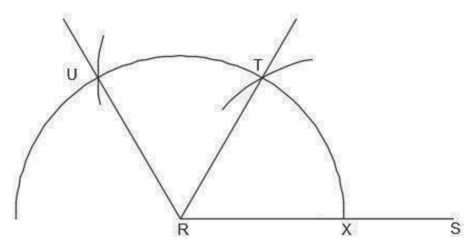
1. Draw the line RS.
2. With centre R, open your compass to any convenient radius and draw a semi-circle that cuts RS at X.
3. With centre X, use the **same radius** and mark another arc on the semi-circle. Label this point T.
4. Draw a line from R to T.



Note that the triangle TRX formed by points in the 60° diagram is equilateral.

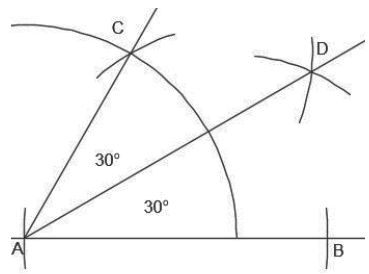
To construct a **120° angle**, continue on the same construction for 60°. Follow these steps:

2. Use the same radius that we used to create the semi-circle.
3. Use T as the centre, and draw another arc on the semi-circle. Label this point U.
4. Draw a line from R to U.



The angles 45°, 30°, and 15° are constructed by simply bisecting angles 60° and 90°. For 15°, the 60° angle will need to be bisected twice. The first bisection gives a 30° angle. Bisection of the 30°angle gives a 15° angle.

For example, these are the steps to construct a **30° angle:**



- Construct a 60° angle using steps from the previous lesson. Label it $\angle CAB$.
- Centre your pair of compasses at the points where the semi-circle intersects CA and AB. Draw arcs from each point, using a convenient radius.
- Label the point where the arcs intersect as D.
- Join A to D to get the angle bisector.

The angles 75°, 105°, and 150° can be constructed using bisection of other angles. They each require you to construct 2 angles in the same diagram and bisect them.

Note that 75° is halfway between 60° and 90°. It is 60° plus 15°. To construct a 75° angle, draw 60° and 90° on the same construction, then bisect the angle between them.

Similarly, 105° is halfway between 90° and 120°. It is 90° plus 15°. To construct a 105° angle, draw 90° and 120° on the same construction, then bisect the angle between them.

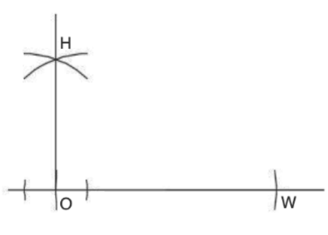
Finally, 150° is halfway between 120° and 180°. It is 120° plus 30°. Recall that 180° is a straight line. To construct a 150° angle, draw 120° and extend the straight line on the same construction. Then, bisect the angle between them.

Solved Examples

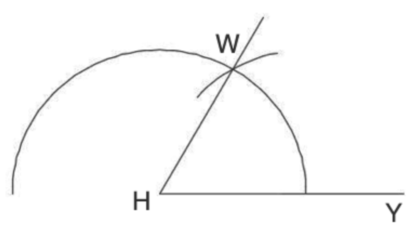
1. Construct the following angles:
    a. An angle of 90°. Label it $\angle HOW$.
    b. An angle of 60°. Label it $\angle WHY$.
    c. An angle of 120°. Label it $\angle POW$.

    Solutions:

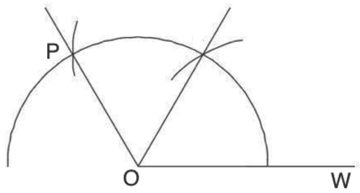
    a.



b.



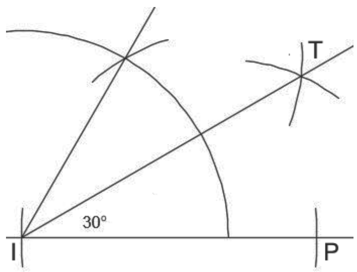
c.



2. Construct the following angles:
    a. An angle of 30°. Label it $\angle TIP$.
    b. An angle of 15°. Label it $\angle MAT$.
    c. An angle of 45°. Label it $\angle BIN$.

    Solutions:

    a.



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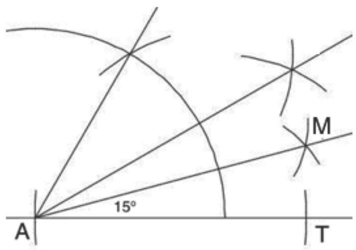
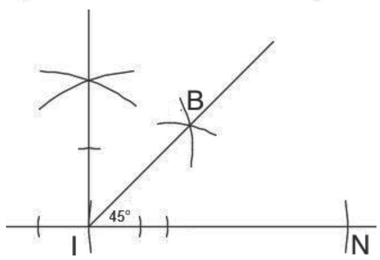


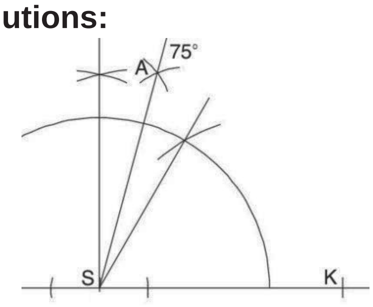
    c.



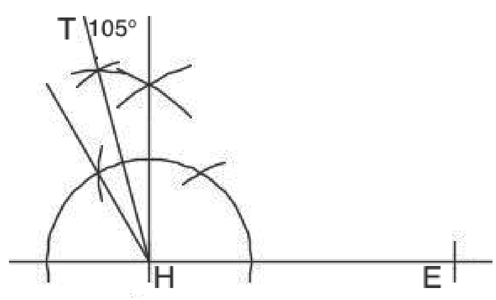
3. Construct the following angles:
    a. An angle of 75°. Label it $\angle ASK$.
    b. An angle of 105°. Label it $\angle THE$.
    c. An angle of 150°. Label it $\angle CAT$.

    Solutions:

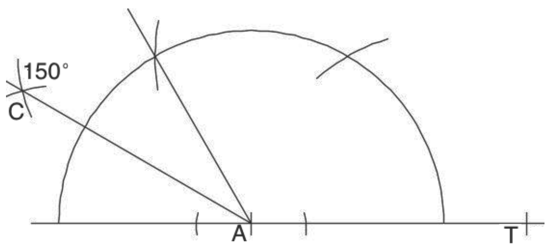
    a.



    b.



c.



4. How could you construct an angle of 165°? Explain in words.

   Solution:

   175° is halfway between 150° and 180°. It could be constructed by constructing an angle of 150°, then bisecting the angle between 150° and 180°.

Practice

1. Construct a 90-degree angle. Label it $\angle CUP$.
2. Construct an angle of 60°. Label it $\angle BIT$.
3. On the same construction as problem 2, construct a 120-degree angle. Label it $\angle LIT$.
4. Construct 45-degree angle. Label it $\angle PAT$.
5. Construct an angle of 30°. Label it $\angle BAT$.
6. On the same construction as problem 2, construction a 15-degree angle. Label it $\angle RAT$.
7. Construct an angle of 75°. Label it $\angle MAP$.
8. Construct an angle of 105°. Label it $\angle FUN$.
9. Construct an angle of 150°. Label it $\angle RUN$.
10. How could you construct an angle of 135°? Explain in words.

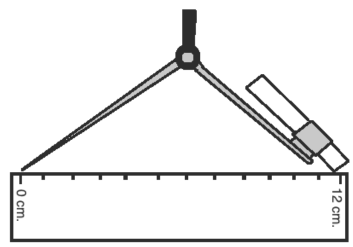
Lesson Title: Triangle construction	**Theme:** Geometry
Practice Activity: PHM4-L080	**Class:** SSS 4

Learning Outcome
By the end of the lesson, you will be able to use a pair of compasses to construct a triangle from given side and angle lengths.

Overview

There are three types of triangle construction problems. You may be asked to construct a triangle given three sides (SSS), with two given sides and an angle (SAS), or with two given angles and a side (ASA).

To construct a triangle given the length of a side, you will need to set the radius of your compass equal to the lengths of the sides of the triangle. Place the tip at zero, and open the pencil's point to the distance you want.



See the diagram at right for how to do this.

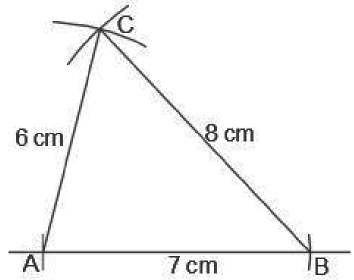
If you have a ruler, use it in class and at home to do construction. If you do not have a real ruler, use the printed one below. This ruler is not exactly to scale (1 cm is not exactly 1 cm). However, it can be used for the purpose of learning construction.

Before starting a construction, it is helpful to quickly sketch a picture of what the constructed triangle will look like. Don't worry about sketching it to scale, this sketch is just to guide your construction.

Consider an example of a SSS problem: Construct a triangle ABC with sides 6 cm, 7 cm, and 8 cm.

To construct ABC, follow these steps:

- Draw a line and label point A on one end.
- Open your compass to the length of 7 cm. Use it to mark point B 7 cm from point A. This gives line segment $\overline{AB} = 7 \text{ cm}$.
- Open your compass to the length of 6 cm. Use A as centre, and draw an arc of 6 cm above $\overline{AB}$.
- Open your compass to the length of 8cm. With the point B as centre, draw an arc that intersects with the arc you drew from point A. Label the point of intersection C.
- Join $\overline{AC}$ and $\overline{BC}$. This is the required triangle ABC.



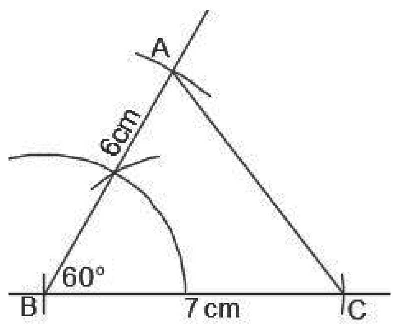
This is the second of 3 lessons on constructing triangles. This lesson uses the lengths of two sides, and the angle between them. These are SAS (side-angle-side) triangles.

To construct a SAS triangle, draw one side with the first known length. Then, construct the angle from this. Extend the constructed line so that it is the second known length. Connect the 2 known sides to make the third side.

Consider an example problem: Construct triangle ABC where $|AB|$ is 6 cm, $|BC|$ is 7 cm, and angle B is 60°.

To construct ABC, follow these steps:

- Draw the side $|BC| = 7 \text{ cm}$ and label it 7 cm.
- From $\overline{BC}$, construct an angle of 60° at B, and label it 60°.
- Open your compass to the length of 6 cm. Use B as centre, and draw an arc of 6 cm on the 60° line. Label this point A.
- Join $\overline{AB}$ and $\overline{BC}$. This is the required triangle ABC.

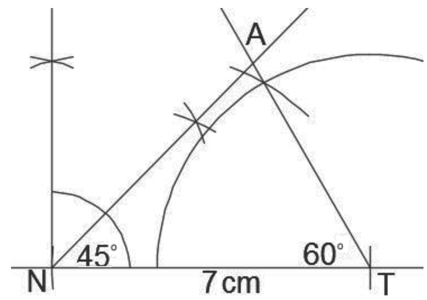


To construct an ASA triangle, draw one side with the known length. Then, construct the 2 given angles on the 2 sides of this. Extend the constructed lines so that they intersect.

Consider an example problem: Construct triangle ANT where $\angle N = 45°$, $\angle T = 60°$ and $\overline{NT} = 7$ cm.

To construct ANT, follow these steps:

- Draw the side $|NT| = 7\text{ cm}$ and label it 7 cm.
- From $\overline{NT}$, construct an angle of 45° at N, and label it 45°.
- From $\overline{NT}$, construct an angle of 60° at T, and label it 60°.
- Extend the 2 angle constructions until they meet. Label this point A. This is the required triangle AN.



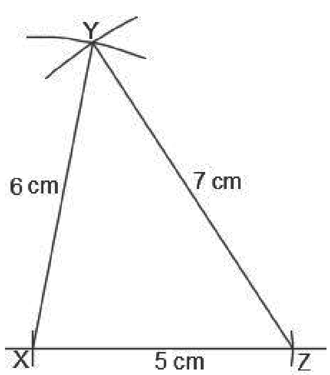
Note that the diagrams in this lesson are not drawn to scale.

Solved Examples

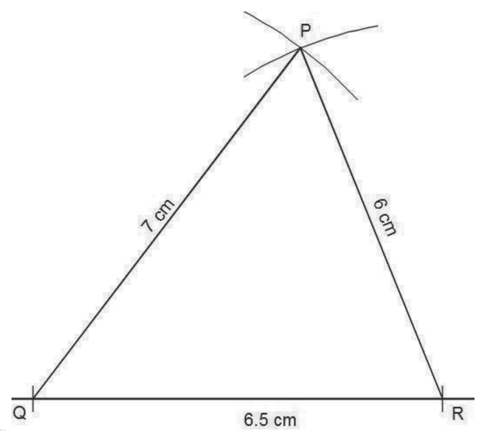
1. Construct the following triangles given the lengths of 3 sides (SSS):
   a. Construct XYZ with sides of length 5 cm, 6 cm, and 7 cm.
   b. Construct a triangle PQR with $|PR| = 6$ cm, $|PQ| = 7$ cm and $|QR| = 6.5$ cm.
   c. Construct a triangle with sides of length 3 cm, 9 cm and 8.5 cm.

   Solutions:

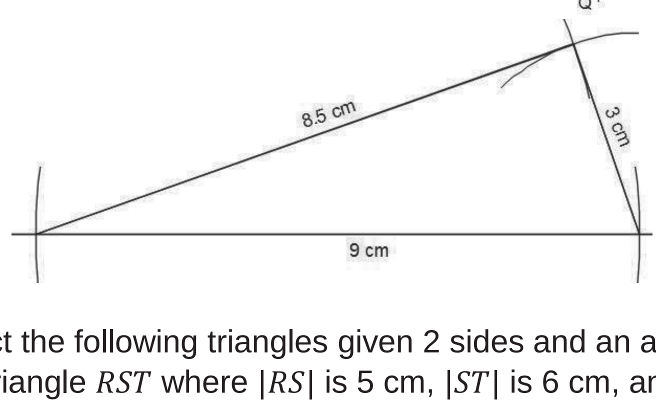
   a.



   b.



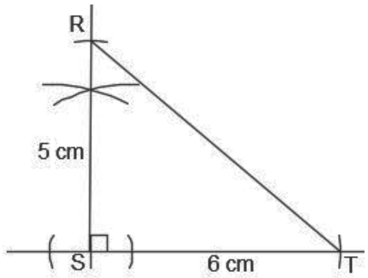
   c.



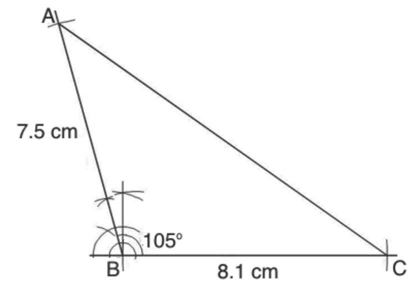
2. Construct the following triangles given 2 sides and an angle (SAS):
   a. Triangle RST where $|RS|$ is 5 cm, $|ST|$ is 6 cm, and $\angle S$ is 90°.
   b. Triangle ABC with $|AB| = 7.5$ cm, $|BC| = 8.1$ cm and $\angle ABC = 105°$.
   c. Construct ΔABC, where $|AB| = 8$ cm, $|BC| = 6$ cm and $\angle ABC = 30°$.

Solutions:

a.



b.



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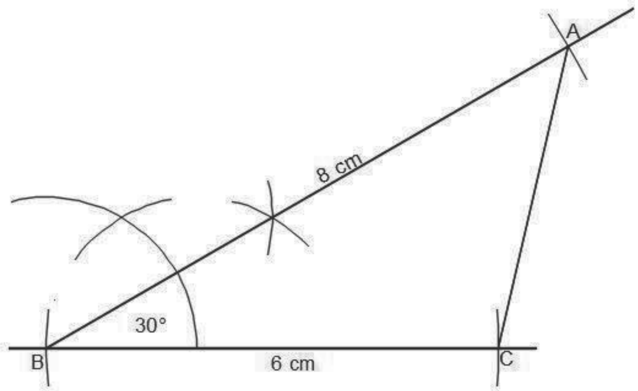
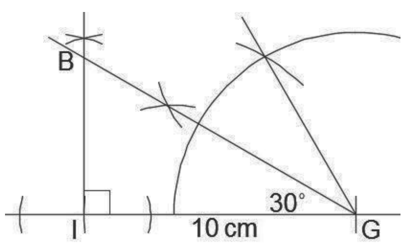


3. Construct the following triangles given 2 angles and a side (ASA):
   a. Triangle BIG where $\angle I = 90°$, $\angle G = 30°$ and $\overline{IG} = 10$ cm.
   b. Construct ΔABC with $|BC| = 6$ cm., $\angle ABC = 30°$ and $\angle ACB = 45°$.
   c. Construct ΔABC with $|AB| = 10$ cm., $\angle ABC = 30°$ and $\angle BAC = 45°$.

   Solutions:

a.



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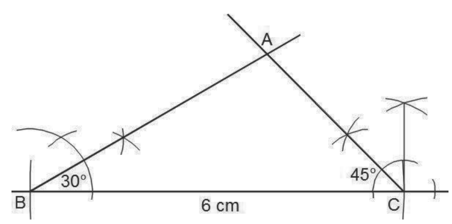
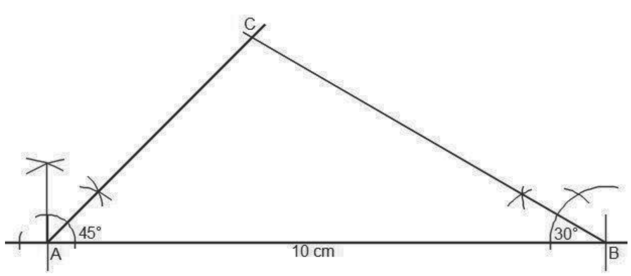


c.



Practice

1. Construct a triangle ABC such that $|AB| = 5$ cm, $|BC| = 6$ cm and $|AC| = 4$ cm.
2. Construct triangle JKL where $|JK|$ is 9 cm, $|KL|$ is 7 cm and $|LJ|$ is 6 cm.
3. Using a ruler and a pair of compasses, construct a triangle ABC such that $|AB| = 6.0$ cm, $|BC| = 8.0$ cm and $|AC| = 7.0$ cm.
4. Using only a ruler and a pair of compasses, construct ΔABC in which $|AC| = 4$ cm, $|BC| = 2$ cm and $\angle C = 75°$.
5. Construct ΔABC in which $|AB| = 7$ cm, $|BC| = 8$ cm and $\angle B = 120°$.
6. Using a ruler and a pair of compasses only, construct ΔCRS with $|RS| = 7$ cm, $\angle R = 45°$ and $\angle S = 90°$.
7. Using a ruler and a pair of compasses only, construct ΔXYZ with $|XY| = 8$ cm, $\angle X = 60°$ and $\angle Y = 30°$.
8. Construct ΔPQR with $|PQ| = 8$ cm, $\angle Q = 90°$ and $\angle P = 30°$.
9. Using a ruler and a pair of compasses only, construct ΔKLM with $|LM| = 7$ cm, $\angle KLM = 105°$ and $\angle KML = 30°$.
10. Using a ruler and a pair of compasses only, construct ΔABC with $|BC| = 6.5$ cm, $\angle ABC = 75°$ and $\angle ACB = 45°$.

Lesson Title: Quadrilateral construction	Theme: Geometry
Practice Activity: PHM4-L081	**Class:** SSS 4

Learning Outcome
By the end of the lesson, you will be able to use a pair of compasses to construct a quadrilateral from given side and angle lengths.

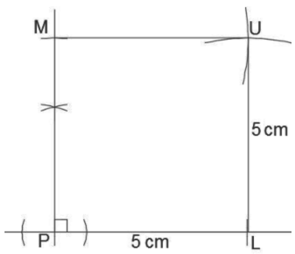
Overview

Quadrilaterals can be constructed using their characteristics. As with triangles, it is always helpful to draw a sketch first.

For **squares and rectangles**, the angles are all right angles, but we only need to construct one of them. After constructing 1 right angle, we can complete the square or rectangle using a pair of compasses to mark the lengths of the sides.

Consider an example problem: Construct a square $PLUM$ with sides of length 5 cm.

To construct $PLUM$, follow these steps:

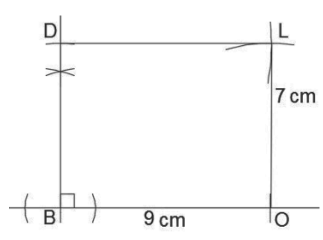


- Draw the side $|PL| = 5 \text{ cm}$ and label it 5 cm.
- From $\overline{PL}$, construct an angle of 90° at P.
- Open your pair of compasses to 5 cm. With P as the centre, draw an arc on the 90° line. Label the intersection M.
- With L as the centre, draw an arc above the line PL. With M as the centre, draw an arc to the right, above L. Label the intersection of these 2 arcs U.
- Draw lines to connect M with U, and U with L.

Rectangle construction follows a similar process, but the pair of compasses should be opened to the appropriate distance to mark the given length and width.

Consider an example: Construct rectangle $BOLD$ where $l = 9$ cm. and $w = 7$ cm.

To construct $BOLD$, follow these steps:



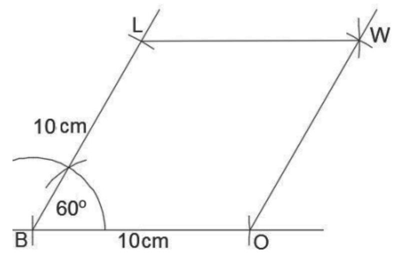
- Draw the side $\overline{BO} = 9 \text{ cm}$ and label it 9 cm.
- From $\overline{BO}$, construct an angle of 90° at B.
- Open your pair of compasses to 7 cm. With B as the centre, draw an arc on the 90° line. Label the intersection D.
- Keep the radius of your pair of compasses at 7 cm. With O as the centre, draw an arc above the line BO.
- Change the radius of your pair of compasses to 9 cm. With D as the centre, draw an arc to the right, above O. Label the intersection of these 2 arcs L.

- Draw lines to connect D with L, and O with L.

Parallelogram and rhombus are constructed in a similar way to each other, because rhombus is a type of parallelogram. In parallelogram construction problems, we are given the measure of an angle, and the lengths of 2 sides. We construct the given angle and extend the sides to the correct lengths.

Consider an example problem: Construct a rhombus $BOWL$ with sides of length 10 cm, and angle $B = 60°$.

To construct $BOWL$, follow these steps:

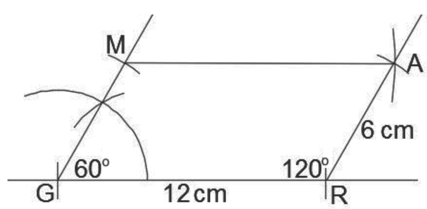


- Draw the side $|BO| = 10 \text{ cm}$ and label it 10 cm.
- From $\overline{BO}$, construct an angle of 60° at B.
- Open your pair of compasses to 10 cm. With B as the centre, draw an arc on the 60° line. Label the intersection L.
- With O as the centre, draw an arc above the line BO. With L as the centre, draw an arc to the right, above O. Label the intersection of these 2 arcs W.
- Draw lines to connect L with W, and W with O.

Parallelogram construction follows a similar process, but the pair of compasses should be opened to the appropriate distance to mark the sides.

Consider the problem: Construct parallelogram $GRAM$ where $|GR| = 12 \text{ cm}$, $|GM| = 6 \text{ cm}$, and angle $G = 60°$.

To construct $GRAM$, follow these steps:



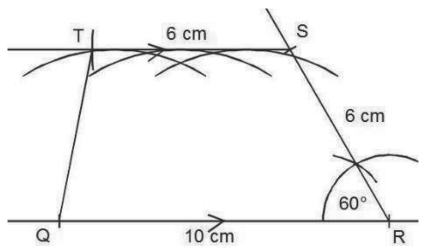
- Draw the side $|GR| = 12 \text{ cm}$ and label it 12 cm.
- From $\overline{GR}$, construct an angle of 60° at G.
- Open your pair of compasses to 6 cm. With G as the centre, draw an arc on the 60° line. Label the intersection M.
- Keep the radius of your pair of compasses at 6 cm. With R as the centre, draw an arc above the line GR.
- Change the radius of your pair of compasses to 12 cm. With M as the centre, draw an arc to the right, above R. Label the intersection of these 2 arcs A.
- Draw lines to connect M with A, and A with R.

In **trapezium** construction problems, we are given the lengths of 3 sides and the measure of at least 1 angle. We are told which 2 sides are parallel.

Consider an example problem: Construct a trapezium $QRST$ such that $|QR| = 10$ cm, $|RS| = 6$ cm, $|ST| = 6$ cm, and $\angle QRS = 60°$ and line $\overline{QR}$ is parallel to line $\overline{ST}$.

To construct $QRST$, follow these steps:

- Draw the side $|QR| = 10$ cm and label it 10 cm.
- From $\overline{QR}$, construct an angle of 60° at R.
- Open your pair of compasses to 6 cm. With R as the centre, draw an arc on the 60° line. Label the intersection S.
- Construct a line parallel to QR:
  - Centre your pair of compasses at S, and open them to the distance between point S and the line QR.
  - Choose any 3 points on line QR. Keep your compass open to the distance between S and QR, and draw 3 arcs above QR.
  - Place your ruler on the highest points of these 3 arcs, and connect them to make a line parallel to QR.

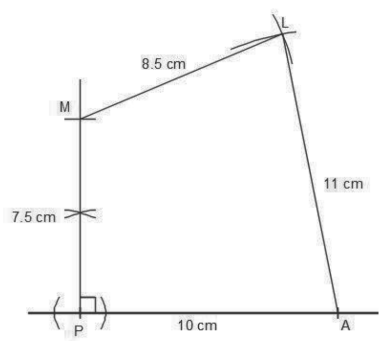


- Open your compass to 6 cm. With S as the centre, draw an arc through the parallel line you constructed. Label the intersection T.
- Draw a line to connect T with Q.

There are **other types of quadrilaterals** that do not fit into a category such as a parallelogram or trapezium. These quadrilaterals can also be constructed if enough information is given.

Consider the following example: Construct quadrilateral $PALM$ where $|PA| = 10$ cm, $|AL| = 11$ cm, $|LM| = 8.5$ cm, and $|MP| = 7.5$ cm. Angle P is a right angle.

To construct $PALM$, follow these steps:



- Draw the side $|PA| = 10$ cm and label it 10 cm.
- From $\overline{PA}$, construct an angle of 90° at P.
- Open your pair of compasses to 7.5 cm. With P as the centre, draw an arc on the 90° line. Label the intersection M.
- Open your compass to 7.5 cm. With M as the centre, draw an arc to the right. Open your compass to 11 cm. With A as the centre, draw an arc above PA. Label the intersection of the two arcs L.
- Draw a line to connect M with L, and a line to connect A with L.

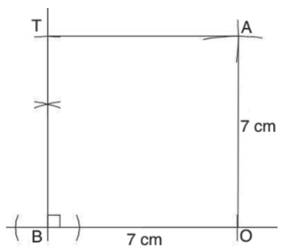
Note that the diagrams in this lesson are not drawn to scale.

Solved Examples

1. Construct the following quadrilaterals:
    a. Square $BOAT$ with sides of length 7 cm.
    b. Rectangle $EFGH$ given the sides $EF = 5$ cm. and $EH = 4$ cm.
    c. Rhombus $COST$ with sides of length 5 cm. and angle $C = 75°$.
    d. Parallelogram, $ABCD$ with $|AB| = 3$ cm., $|AD| = 5$ cm. and $\angle DAB = 45°$.
    e. Trapezium $BASK$ where $|BA| = 15$ cm., $|BK| = 9$ cm., $|SK| = 10$ cm., and $\angle B = 60°$ and line $\overline{KS}$ is parallel to line $\overline{BA}$.
    f. Quadrilateral $ABCD$ such that $|AB| = 5$ cm, $|BC| = 7.6$ cm, $|CD| = 4$ cm and $|DA| = 5.5$ cm and $\angle A = 75°$.

Solutions:

a.



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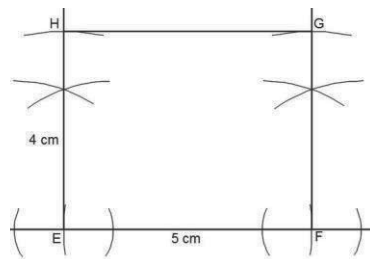

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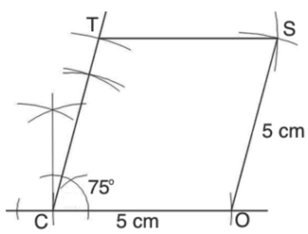

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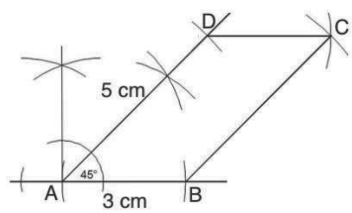

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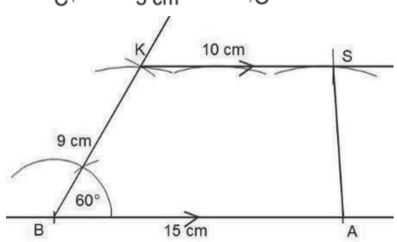
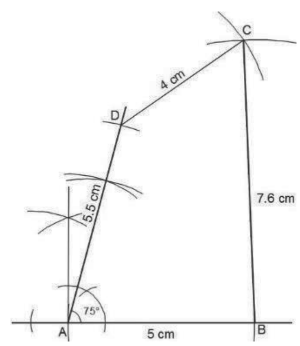


f.



Practice

1. Construct the following squares:
    a. $EACH$ where $|EA| = 5$ cm and $|EH| = 5$ cm.
    b. $RATE$ with sides of length 6 cm.
2. Construct the following rectangles:
    a. $PINT$ where $|PI| = 6$ cm and $|PT| = 4$ cm.
    b. $COKE$ where $|CO| = 7$ cm and $|CE| = 2.5$ cm.
3. Construct rhombus $GIRL$ with sides of length 7 cm and angle $I = 60°$.

4. Construct rhombus $CITE$ with sides of length 5.5 cm and angle $C = 45°$.
5. Construct parallelogram $PQRS$ such that $|PQ| = 7$ cm, $|QR| = 4.5$ cm and $\angle SPQ = 135°$.
6. Construct parallelogram $ABCD$ such that $|AB| = 6$ cm, $\angle B = 105°$ and $|AD| = 4$ cm.
7. Using a ruler and a pair of compasses only, construct quadrilateral $ZINC$ such that $|ZI| = 6 \text{ cm}$, $|IN| = 5 \text{ cm}$, $|NC| = 4 \text{ cm}$, and $\angle I = 45°$.
8. Using a ruler and a pair of compasses only, construct trapezium $PQRS$ such that $|PQ| = 8 \text{ cm}$, $|PS| = 6 \text{ cm}$, $|RS| = 6 \text{ cm}$, $\angle P = 75°$, and SR is parallel to PQ.
9. Using a ruler and a pair of compasses only, construct trapezium $BUST$ such that $|BU| = 12 \text{ cm}$, $|BT| = 6 \text{ cm}$, $|ST| = 5 \text{ cm}$, $\angle B = 30°$, and line TS is parallel to BU.
10. Using a ruler and a pair of compasses only, construct quadrilateral $WXYZ$ such that $|WX| = 5.5 \text{ cm}$, $|XY| = 5.8 \text{ cm}$, $|YZ| = 4.2 \text{ cm}$, $|ZW| = 7.8 \text{ cm}$ and angle $\angle W = 45°$.

Lesson Title: Construction of loci	Theme: Geometry
Practice Activity: PHM4-L082	**Class:** SSS 4

Learning Outcome
By the end of the lesson you will be able to use a pair of compasses to construct various loci.

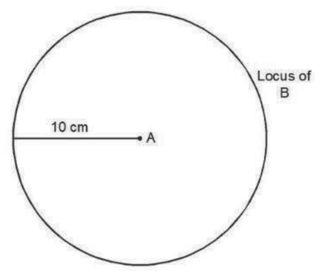
Overview

This lesson focuses on 4 different ways of constructiong loci (the plural of locus). A locus is a specific path that a point moves through. The point obeys certain rules as it moves through the locus. These are the 4 types of loci that you will construct:

1. The locus of a point P from a given distance from a point.
2. The locus of a point P equidistant from 2 given points.
3. The locus of a point P equidistant from 2 given lines.
4. The locus of a point P a given distance from a line segment and a line.

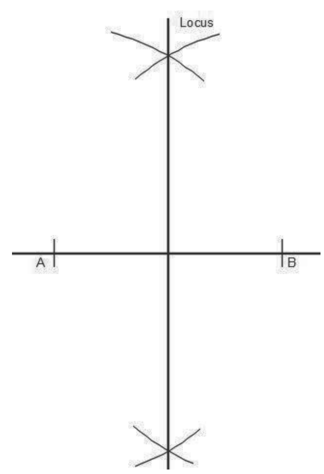
Note that the diagrams in this lesson are not drawn to scale.

The **locus of points that is a given distance from a given point** is a circle. For example, consider the locus of point B if it is 10 cm from point A. The locus of point B is all of the possible points where B could exist. The locus of points where B could exist is a circle with radius 10 cm, with centre A. The locus of B is shown to the right.



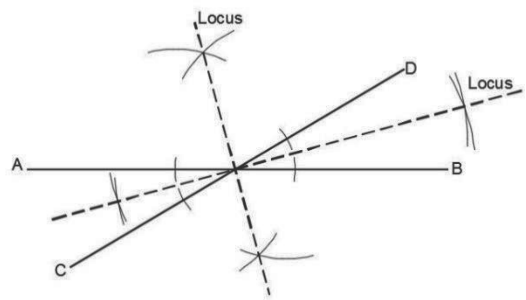
The **locus of points that is equidistant from 2 given points** is the perpendicular bisector of the line that connects the 2 given points.

For example, consider two points A and B. The locus of points that is equidistant from these 2 points is a vertical line between them. This line is the perpendicular bisector of line AB. Construct the perpendicular bisector to show the locus of points equidistant from A and B (shown on the right).



The **locus of points that is equidistant from 2 given intersecting lines** is found by bisecting the angles formed by the lines.

For example, consider two lines AB and CD. The locus of points that is equidistant from these 2 lines is the set of angle bisectors. Construct each angle bisector to show the locus of points equidistant from AB and CD (shown on the right).



The **locus of points that is a given distance from a given line segment** is an oblong shape around the line. For example, consider the problem: Construct the locus of points 6 cm from line segment AB.

Follow these steps to draw the construction:

- Open your pair of compasses to a radius of 6 cm.
- Taking both A and B as the centre, draw semi-circles left and right of the line:

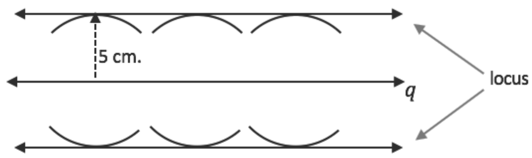
- Use a straight edge to connect the semi-circles above and below the line:

Recall that a line can extend in 2 directions forever, which is shown with arrows at the ends of the line. For such lines, **the locus of points a given distance from the line is 2 additional, parallel lines**.

For example, consider the problem: Construct the locus of points 5 cm from line q.

Follow these steps to draw the construction:

- Open your pair of compasses to a radius of 5 cm.
- Choose several points on q, and centre your compass at each. From each point, draw an arc directly above and below line q.
- Hold a straight edge along the points of the arcs farthest from line q. Connect these points.
- Draw arrows to show that the locus extends forever in both directions.

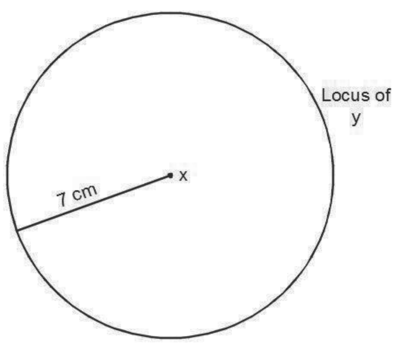


Solved Examples

1. Construct loci for the following problems:
    a. The locus of a point (y) moves so that it is 7 cm away from a fixed point (x). Construct the locus of y.
    b. Points C and D are 14 cm from one another. Point E is equidistant from C and D. Draw the locus of point E.
    c. Lines XY and CD intersect at point O such that $\angle COY = 60°$. Construct the lines, then construct the locus of a point P that is equidistant from the 2 lines.
    d. Draw line $XY = 6cm$. Construct the locus of a point P that is 2 cm from the line.
    e. Draw a line J that extends forever in both directions. Construct the locus of a point if it is 5 cm from J.

Solutions:

a.



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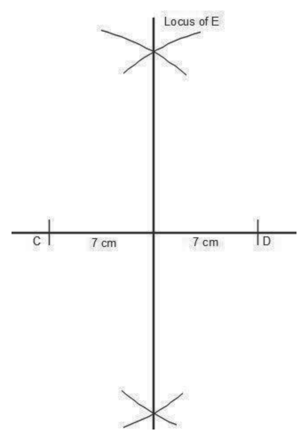

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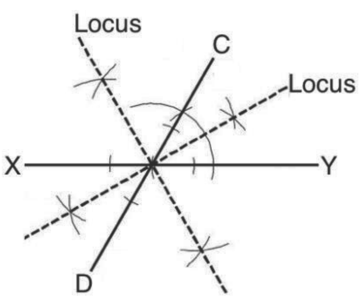

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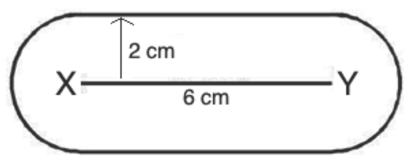

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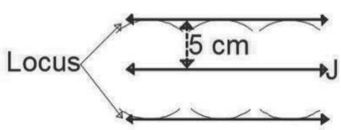
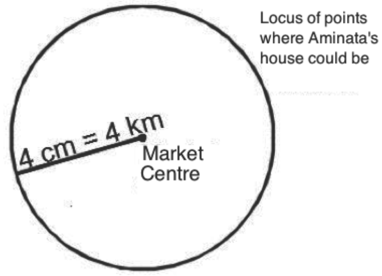
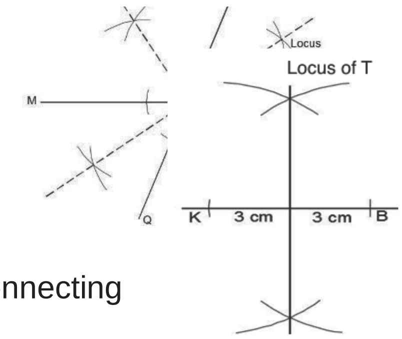


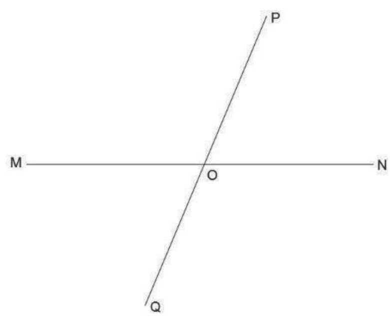
2. Aminata, a trader lives exactly 4 km from the market centre. Some business partners are looking for her house. Draw the locus of all possible locations of Aminata's house. Use 1 m for each kilometre.
   Solution:
   The solution is a circle with a radius of 4 cm and the market in the centre.



3. The distance between Mr. Kamara's house and Mr. Banguara's house is 60 km. Mr. Turay's house is equidistant from from Mr. Kamara and Mr. Bangura's house. Draw the locus of Mr. Turay's house. Use a scale of 1 cm to 10 km.
   Solution:
   The solution is a perpendicular bisector of the line connecting Mr. Kamara's house to Mr. Bangura's house.



4. On the lines LM and PQ below, construct the locus of point equidistant form the 2 lines.



   Solution:
   Your construction above should look like this:

Practice

1. A goat is tied to a rope that measures 30 metres in length. It is tied to a pole and feeding on the grass. Construct the locus of points where the goat feeds. Use a scale of 1 cm to 10 metres.
2. A is a point 4 cm from x. Draw the locus of A.
3. A dog is tied to a chain that measures 3 metres long, and keeps on moving around wishing for freedom. It walks around in a circle. Draw the locus of the dog. Use a scale of 1 cm for each metre.

4. Musa and Idrissa are standing 40 metres from one another. Their classmate John is walking in a straight line. He maintains the same distance from Musa and Idrissa. Construct John's path. Use a scale of 1 cm to 10 metres.
5. T is a point that is equidistant from 2 points X and Y. Draw X and Y a distance of 5 cm from each other. Construct the locus of T.
6. Lines WX and YZ intersect at point O such that $\angle YOX = 45°$. Construct the lines, then construct the locus of a point P that is equidistant from the 2 lines.
7. Draw any 2 intersecting lines and label them AB and CD. Construct the locus of a point P that is equidistant from 2 lines.
8. Andrew owns a piece of land with sides that form a $60°$ angle. He wants to erect a fence equidistant from the 2 lines.
   a. Construct 2 sides of the land.
   b. Construct the locus of points where he could erect the fence.
9. Draw a line segment $PQ = 7$ cm. Construct the locus of points 2.5 cm from the line segment.
10. Draw a line V that extends in both directions. Construct the locus of points 10 mm from the line.

Lesson Title: Construction word problems	**Theme:** Geometry
Practice Activity: PHM4-L083	**Class:** SSS 4

Learning Outcome
By the end of the lesson, you will be able to construct shapes based on information given in word problems.

Overview

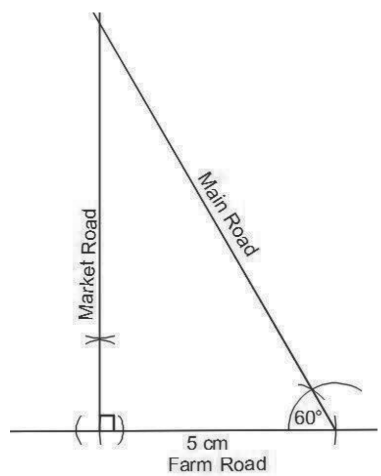
Construction can be used in many real-life situations. This lesson uses information from previous lessons to construct shapes from word problems.

Solved Examples

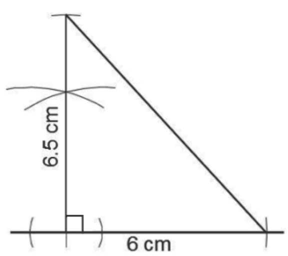
1. Three roads intersect each other to form a triangle. Market Road and Farm Road intersect at a 90° angle. Five kilometres from that intersection, Farm Road intersects with Main Road at a 60° angle. Construct the triangle formed by the 3 roads. Use 1 cm for each kilometre.

   Solution:

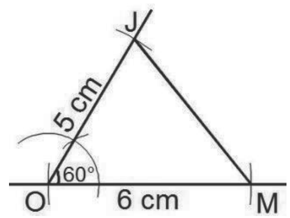
   Two angles (90°, 60°) and a side (5 km) are known. We will construct an ASA triangle. It is helpful to first draw a sketch. Then, use a pair of compasses to construct the triangle.



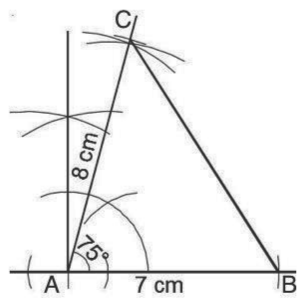
2. Jim, a contractor, wants to construct a floor in a triangular shape. It will have a base of length 6 metres, and a height of 6.5 metres. The base and height will meet at a 90-degree angle. Construct the shape of the floor, using 1 cm for each metre.
   Solution:



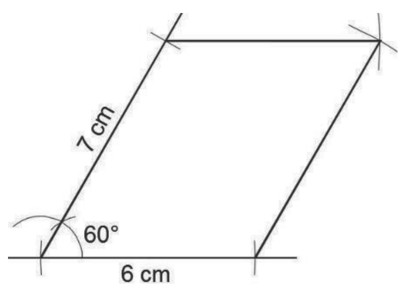
3. Mary and Jane live in different houses in a city. On a map, their roads form a triangle, intersecting at a $60°$ angle. On their way to school, they meet at that intersection. If the distance covered by Mary is 60 metres and Jane 50 metres, construct the triangle. Use 1 centimetre for 10 metres.
Solution:



4. Two vehicles leave town A at the same time. One vehicle travels 70 km from town A to town B. The other vehicle travels 80 km to town C. Their paths form a $75°$ angle. Construct a triangle showing towns A, B, and C. Use a scale of 1 cm to 10 km.
Solution:



5. A farmer went to build a pottery house in the shape of a parallelogram. The building has adjacent sides of 6 and 7 metres, which join at an angle of $60°$. Construct the shape of the building using a scale of 1 cm for every metre.
Solution:

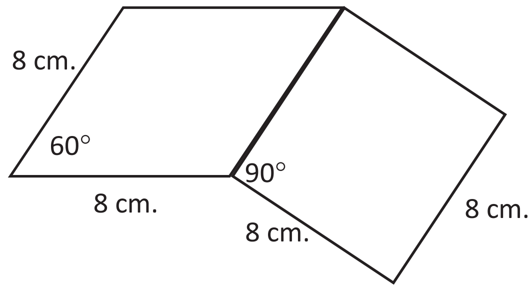


6. Mohamed has a large piece of land. He planted a garden in the shape of a rhombus with sides of length 80 m and one angle of 60°. He wants to plant more crops. He decides to plant an additional plot in the shape of a square. The square shares a side with the rhombus. Construct the shape of Mohamed's garden. Use 1 cm for each 10 m.

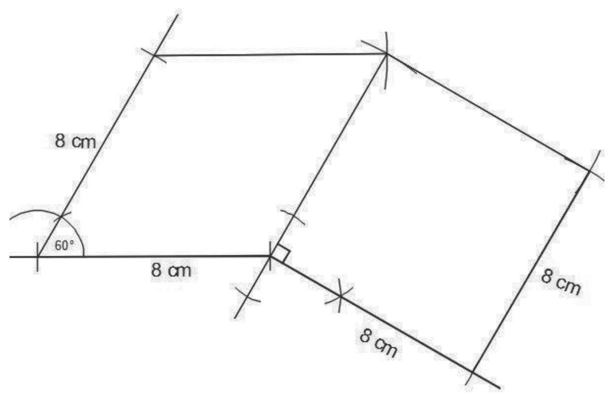
Solution:

Using 1 cm for each 10 m, each side of the rhombus will be 8 cm long. Since the square shares a side with the rhombus, its sides will also be 8 cm long.

Draw a sketch of the shape:



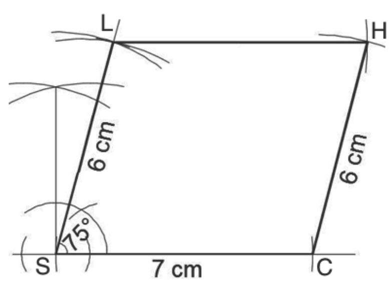
Construct the shape. Start with the rhombus, then construct the square using any side of the rhombus.



Note that the diagram may look different depending on which side the square is drawn on.

7. The principal of Happiness Secondary School wants to draw an accurate map of the school. The school's land is in the shape of a parallelogram. He labels the corners $SCHL$, and measures side SC to be 70 metres and CH to be 60 metres. He also knows that the measure of corner S is $75°$. Construct the boundary of the school's land, using a scale of 1 cm for 10 metres.

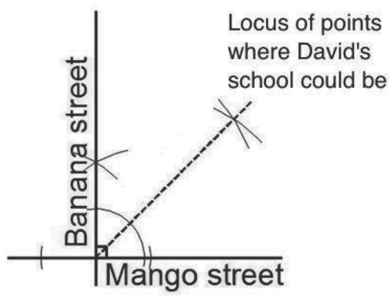
   Solution:



8. David's school is situated at a junction of two streets, Banana Street and Mango Street. The intersection of the streets forms a $90°$ angle, and the school is equidistant from the 2 streets.
    a. Draw the intersection of the two streets.
    b. Construct the locus of points where the school could be.

Solutions:

a. and b.:



Practice

1. Two helicopters leave from airport K. The first helicopter travels 400 kilometres to airport M. The second helicopter travels 800 kilometres to airport L. Their paths form an angle of 60°. Construct a triangle to show the relationship between the 3 airports. Use 1 cm for 100 km.
2. A carpenter cuts a triangular piece of wood. It has sides of length 90 cm and 65 cm, which form a 45° angle. Construct the triangle, using a scale of 1 cm for 10 cm.
3. Abu went to the farm for an agricultural practical. He was to construct a seed bed, which took the shape of an equilateral triangle with sides 8 metres in length. Construct the shape of the seed bed using a scale of 1 cm for each metre.
4. Alpha, a carpenter, is to make a table top that is rectangular in shape and measures 80 centimetres long and 40 metres wide. Construct the shape of the table top using 1 cm for each 10 cm.
5. Ansu, a contractor, constructed a building. Its front is a rectangle 10 metres long and 5 metres tall. He further constructed a roof in the shape of a triangle that forms a $30°$ angle at the top of each side of the rectangle. Construct the shape of the face of the house. Use 1 cm for each metre.

Lesson Title: Construction of complex shapes	**Theme:** Geometry
Practice Activity: PHM2-L084	**Class:** SSS 4

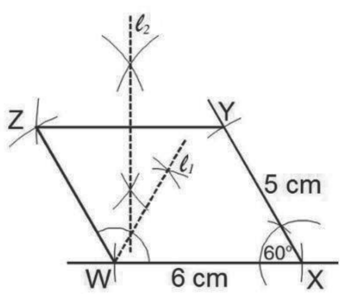
Learning Outcome
By the end of the lesson, you will be able to use a pair of compasses to construct various complex shapes.

Overview

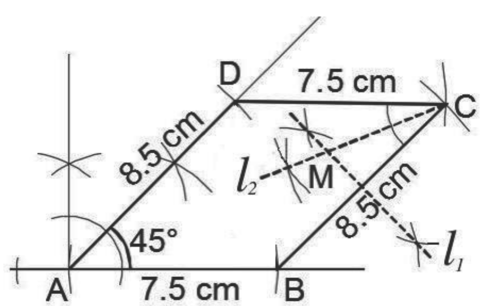
This lesson combines the information you learned in the previous lessons on geometry construction. You will be constructing shapes and loci in the same diagram.

Solved Examples

1. Using a ruler and a pair of compasses only, construct:
   a. Parallelogram $WXYZ$ such that $\angle X = 60°$, $|WX| = 6$ cm and $|XY| = 5$ cm.
   b. The locus l_1 of points equidistant from XW and WZ.
   c. The locus l_2 of points equidistant from Y and Z.
   Solutions:

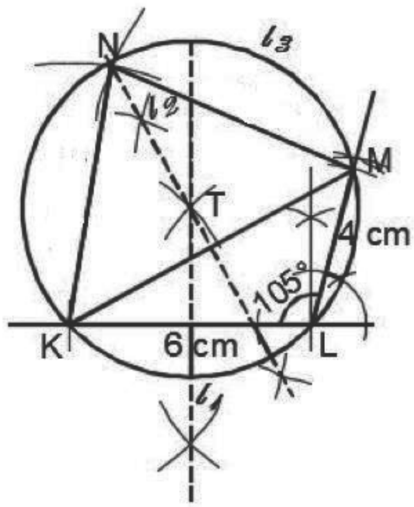


2. Using a ruler and a pair of compasses only:
   a. Construct a parallelogram $ABCD$ such that $|AB| = 7.5$ cm, $|AD| = 8.5$ cm, $\angle DAB = 45°$ and $AD||BC$.
   b. Construct:
      i. Locus l_1 of points equidistant from B and C.
      ii. Locus l_2 of points equidistant from CD and CB.
   c. Locate M, the point of intersection of l_1 and l_2.
   Solutions:



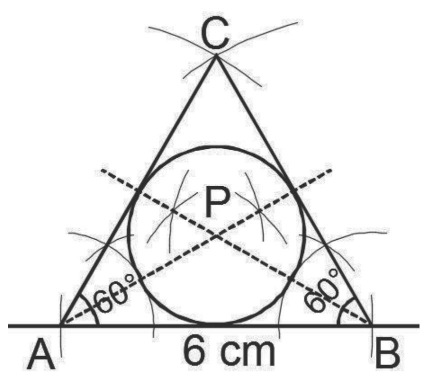
3. Using a ruler and a pair of compasses only, construct:
   a. $\Delta\, KLM$ such that $|KL| = 6$ cm, $|LM| = 4$ cm and $\angle KLM = 105°$.
   b. Locus l_1 of points equidistant from K and L.
   c. Locus l_2 of points equidistant from K and M.
   d. Label the point T and where l_1 and l_2 intersect.
   e. With centre T and radius $|TK|$, construct a circle l_3.
   f. Complete quadrilateral $KLMN$ such that N lies on the circle and $|KN| = |MN|$.

   Solutions:



4. Using a ruler and a pair of compasses only,
   a. Construct $\Delta\, ABC$ such that $|AB| = 6$ cm and $\angle A = \angle B = \angle C = 60°$.
   b. Locate a point P inside the triangle equidistant from AB and AC, and also equidistant from BA and BC.
   c. Construct a circle with centre at P.

   Solutions:



Practice

1. Using a ruler and a pair of compasses only, construct:
    a. A quadrilateral $WXYZ$ with $|WX| = 5.5$ cm, $\angle YXW = 90°$, $|YX| = 7$ cm, $|YZ| = 8$ cm and $|WZ| = 6$ cm.
    b. The bisectors of $\angle W$ and $\angle X$ to meet at O.
2. Using a ruler and a pair of compasses only, construct:
    a. Quadrilateral $ABCD$ such that $|AB| = 8$ cm, $|AD| = 7$ cm, $|BC| = 5$ cm, $\angle DAB = 60°$ and $\angle ABC = 75°$.
    b. The locus l_1 of points equidistant from AD and CD.
    c. The locus l_2 of points equidistant from C and D
3. Using a ruler and a pair of compasses only, construct:
    a. Construct triangle XYZ such that $|XY| = 7$ cm, $|XZ| = 6$ cm and $\angle YXZ = 75°$.
    b. Construct the locus l_1 of points equidistant from X and Z.
    c. Construct the locus l_2 of points equidistant from X and Y.
    d. Locate the point O equidistant from X, Y and Z.
    e. With O as centre, draw the circle l_3 that circumscribes the triangle XYZ.
4. Using a ruler and a pair of compasses only, construct:
    a.
        i. A quadrilateral PQRS, where |PQ|= 10 cm, |PS|= 8 cm, |QR|= 12 cm, $\angle QPS = 60°$, and $\angle PSR = 135°$.
        ii. The locus, l_1 of points equidistant from QR and RS.
        iii. The line, l_2, from Q perpendicular to l_1.
    b.
        i. Locate M, the point of intersection of l_1 and l_2.
        ii. Measure |PM| and |SM|.
5. Using a ruler and a pair of compasses only, construct:
    a.
        i. A quadrilateral MNOP such that |MN|= 7.5 cm, |NO|= 6 cm, |MP|= 5 cm, $\angle MNO = 60°$ and $\angle NMP = 75°$.
        ii. The locus l_1 of points equidistant from M and N.
        iii. The locus l_2 of points equidistant from MN and MP.
    b. Locate a point E, where E is the point of intersection of l_1 and l_2.
    c. Measure |ME| and |EP|.
6. Using a ruler and a pair of compasses only, construct:
    a. Triangle XYZ with |XY|= 8.5 cm, |YZ|= 9.1 cm, and $\angle XYZ = 105°$.
    b. Locate a point P on YZ such that |YP| : |PZ|= $4:3$.
    c. Through P, construct a line l perpendicular to YZ.
    d. If the line l meets XZ at M, measure |YM|.

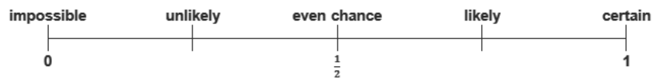
Lesson Title: Addition law of probability	Theme: Probability and Statistics
Practice Activity: PHM4-L085	**Class:** SSS 4

Learning Outcome
By the end of the lesson, you will be able to apply the addition law to find the probability of mutually exclusive events.

Overview

Review of Probability

Probabilities are given values between 0 and 1. A probability of 0 means that the event is impossible, and a probability of 1 means that it is certain. This is shown on the scale:



For equally likely outcomes, the probability that an event, E, will happen is:

$$P(E) = \frac{\text{number of ways of obtaining event } E}{\text{total number of possible outcomes}}$$

As an example, when tossing a fair coin, the probability of getting heads is:

$$P(\text{head}) = \frac{\text{heads on a coin}}{\text{all possible outcomes}} = \frac{1}{2}$$

The probabilities of all possible outcomes sum to one. Considering a coin again,

$$P(\text{head}) + P(\text{tail}) = \frac{1}{2} + \frac{1}{2} = 1$$

or, $P(\text{head}) + P(\text{not head}) = 1$

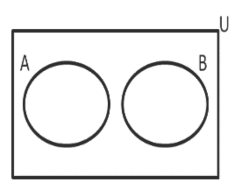
This gives the equation $P(\text{not head}) = 1 - P(\text{head})$. This is called the **complement of the event.** If the probability of obtaining a head is denoted as $P(H)$, the complement is written as $P(\overline{H})$.

Addition Law of Probability

If two events cannot happen at the same time, then they are called **mutually exclusive events**. For example, you cannot win a game and lose a game at the same time. Mutually exclusive events are examples of compound or combination events. The events are connected by the word "or".

If two events A and B are mutually exclusive events, then the probability of A or B is given by:

$$\begin{aligned} P(A \text{ or } B) &= P(A \cup B) \\ &= P(A) + P(B) \end{aligned}$$



This is the Addition Law for mutually exclusive events. In probability, the word "or" or the symbol ∪ indicates addition. A and B are disjoint sets as shown by the Venn diagram.

For two mutually exclusive events which cover all possible outcomes, all the individual probabilities add up to 1: $P(A) + P(B) = 1$

This applies to cases where there are more than 2 events as well:

$$\begin{aligned} P(A \text{ or } B \text{ or } C \text{ or } D \text{ or } \ldots) &= P(A) + P(B) + P(C) + P(D) + \cdots \\ P(A) + P(B) + P(C) + P(D) + \cdots &= 1 \end{aligned}$$

Solved Examples

1. A card is taken at random from an ordinary pack of cards. What is the probability that it will be an Ace or the 10 of Clubs?

   Solution:

   Step 1. Find the individual probabilities.

   the total number of possible outcomes $n(S) = 52$

   probability of an event E occurring $P(E) = \frac{n(E)}{n(S)}$

   Let A be the event of choosing an ace, B the event of 10 of clubs

   $A = \{$ace of clubs, ace of spades, ace of diamonds, ace of hearts$\}$

   $n(A) = 4 \qquad P(A) = \frac{4}{52} = \frac{1}{13}$

   $B = \{$10 of clubs$\}$

   $n(B) = 1 \qquad P(B) = \frac{1}{52}$

   Step 2. Find the probability of Ace or 10 of Clubs.

   $$P(A \text{ or } B) = P(A) + P(B) = \frac{1}{13} + \frac{1}{52} = \frac{4}{52} + \frac{1}{52} = \frac{5}{52}$$

2. The word P R O B A B I L I T Y was written on identical pieces of paper and put in a bag. One of the pieces of paper is selected at random. What is the probability of getting:

   a. A   b. B   c. I
   d. A or B   e. B or I   f. A or B or I

   Solutions:

   First, note the number of possible outcomes: $n(S) = 11$, the number of letters in the bag. This will be the denominator of each fraction.

   a. Since $n(A) = 1$, the probability of getting A is $P(A) = \frac{1}{11}$
   b. Since $n(B) = 2$, the probability of getting B is $P(B) = \frac{2}{11}$
   c. Since $n(I) = 2$, the probability of getting I is $P(I) = \frac{2}{11}$
   d. Add $P(A)$ and $P(B)$:

$P(A \text{ or } B) = P(A) + P(B) = \frac{1}{11} + \frac{2}{11} = \frac{3}{11}$

e. Add $P(B)$ and $P(I)$:

$P(B \text{ or } I) = P(B) + P(I) = \frac{2}{11} + \frac{2}{11} = \frac{4}{11}$

f. Add $P(A)$, $P(B)$ and $P(I)$:

$P(A \text{ or } B \text{ or } I) = P(A) + P(B) + P(I) = \frac{1}{11} + \frac{2}{11} + \frac{2}{11} = \frac{5}{11}$

3. A bag contains a number of balls of different colours. The probability of obtaining a ball of a particular colour is given in the table below.

Colour	Probability
black	$\frac{3}{8}$
white	$\frac{1}{4}$
yellow	$\frac{1}{5}$

What is the probability that a ball taken at random from the bag is:

a. Black or white

b. Not white or yellow

c. Not one of the colours listed in the table

Solutions:

Note that the number of possible outcomes is $n(S) = 3$.

Let the initials of the colours represent their respective events.

a. $P(B \text{ or } W) = P(B) + P(W) = \frac{3}{8} + \frac{1}{4} = \frac{5}{8}$

b. $\overline{W}$ is the complement of W, or the probability that a ball drawn is not white. Likewise, $\overline{Y}$ is the complement of Y. We have:

$P(\overline{W \text{ or } Y}) = P(\overline{W} \text{ or } \overline{Y}) = 1 - (P(W) + P(Y))$

Find $P(W) + P(Y)$:

$P(W) + P(Y) = \frac{1}{4} + \frac{1}{5} = \frac{9}{20}$

Therefore, $P(\overline{W} \text{ or } \overline{Y}) = 1 - \frac{9}{20} = \frac{11}{20}$

c. To find the probability that a ball is none of the colours listed, subtract the probabilities of all of the given colours from 1.

$$\begin{aligned} P(\text{none of colours listed}) &= 1 - P(\text{one of the colours listed}) \\ &= 1 - (P(B) + P(W) + P(Y)) \\ &= 1 - \left(\frac{3}{8} + \frac{1}{4} + \frac{1}{5}\right) \\ &= 1 - \frac{33}{40} \\ &= \frac{7}{40} \end{aligned}$$

Practice

1. Which of these pairs of events are mutually exclusive?
   a. Studying Mathematics and studying Geography.
   b. Choosing an even number and a prime number less than 10.
   c. Eating garri for breakfast and rice for lunch.
   d. Getting the right answer and the wrong answer for the same Maths problem.
2. A boy picked a number from the integers 10 to 25 inclusive. What is the probability that it is either a prime or an even number?
3. If a number is chosen at random from the integers 5 to 25 inclusive, find the probability that the number is:
   a. A multiple of 3 or 10.
   b. An even or prime number.
   c. Less than 12 or greater than 18.
4. A bag contains 10 balls. Six of the balls are red and the rest are equal numbers of white and blue. If a ball is picked at random. Find the probability that it is:
   a. Either white or red
   b. Not red
   c. Blue or red
   d. White
5. The table shows the probability of getting a particular colour of counters which have been put in a bag. There are only 4 colours in the bag.

Colour	Yellow	Red	Green	Blue
Probability	0.5	0.2		0.1

   a. Complete the table to show the probability of getting green.
   b. A counter is taken at random from the bag, what is the probability of getting yellow or blue?

Lesson Title: Multiplication law of probability	Theme: Probability and Statistics
Practice Activity: PHM4-L086	**Class:** SSS 4

Learning Outcome
By the end of the lesson, you will be able to apply the multiplication law to find the probability of independent events.

Overview

If one event happening has no effect on another event happening they are called **independent events**. For example, being a girl and being left-handed are independent events. Being a girl does not have any effect on which hand is used to write and being left-handed does not have an effect on being a girl. Independent events are examples of compound or combination events. The events are connected by the word "and".

Other examples of independent events include: raining on Monday this week, raining on Monday next week; a coin tossed twice lands on head then lands on tail; a die rolled twice shows a 6 then an odd number.

If two events A and B are independent events, then the probability of A and B is given by:

$$\begin{aligned} P(A \text{ and } B) &= P(A \cap B) \\ &= P(A) \times P(B) \end{aligned}$$

This is the Multiplication Law for independent events. In probability, the word "and" or the symbol ∩ indicates multiplication.

This applies to cases where there are more than 2 events as well:

$$\begin{aligned} P(A \text{ and } B \text{ and } C \text{ and } D \text{ and } \ldots) &= P(A \cap B \cap C \cap D \ \ldots) \\ &= P(A) \times P(B) \times P(C) \times P(D) + \cdots \end{aligned}$$

As before, $P(\text{not } A) = 1 - P(A)$

Solved Examples

1. A fair die is rolled twice. What is the probability that it will land on a 6 in the first roll and land on an odd number in the second roll?
   Solution:
   Step 1. Find the possible outcomes.
   The possible outcomes are $S = \{1, 2, 3, 4, 5, 6\}$, so the total number of possible outcomes is $n(S) = 6$.
   Step 2. Find the probability of each independent event:
   Probability of rolling a 6: $P(6) = \frac{1}{6}$

Probability of rolling an odd number: $P(\text{odd number}) = \frac{3}{6} = \frac{1}{2}$

Step 3. Find the probability of rolling a 6 **and** rolling an odd number:

$P(6 \text{ and odd number}) = P(6) \times P(\text{odd number}) = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$

2. A coin is tossed and a die is rolled. What is the probability of getting a head on the coin and an even number on the die?

Solution:

The possible outcomes of the coin are: $S_{\text{coin}} = \{H, T\}$ so that $n(S_{\text{coin}}) = 2$.

The possible outcomes of the die are: $S_{\text{die}} = \{1, 2, 3, 4, 5, 6\}$ so $n(S_{\text{die}}) = 6$.

Probability of getting a head: $P(H) = \frac{1}{2}$

Probability of getting an even number: $P(\text{even}) = \frac{3}{6} = \frac{1}{2}$

Multiply to find the probability of getting a head **and** even number:

$$P(H \text{ and even}) = P(H) \times P(\text{even}) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

3. A bag contains 4 green balls and 6 red balls. A ball is picked from the bag without replacement, a second ball is then picked from the bag. Find the probability that:
    a. The first is red and the second is also red.
    b. The first is green and the second is also green.
    c. The first is red and the second is green.
    d. The first is green and the second is red.

Solutions:

The total number of balls in the bag is 10. When a ball is picked out without a replacement, the total number of balls for the second event is 9.

a. Probability that the first is red and the second is also red: $\frac{6}{10} \times \frac{5}{9} = \frac{1}{3}$

Note that after picking the first red ball, the number of red balls remaining is 5 and the total number of balls remaining in the bag is 9 so the probability of picking a second red is $\frac{5}{9}$.

b. Similarly, the probability that the first is green and the second is also green is $\frac{4}{10} \times \frac{3}{9} = \frac{2}{15}$.

c. Probability of first red and second green: $\frac{6}{10} \times \frac{4}{9} = \frac{4}{15}$

d. Probability of first green and second red: $\frac{4}{10} \times \frac{6}{9} = \frac{4}{15}$

4. A ball is picked at random from each of two bags A and B. Bag A contains 5 blue balls and 3 white balls, and bag B contains 4 blue balls and 8 white balls. What is the probability of getting:
    a. A blue ball from bag A and a blue ball from bag B.
    b. A white ball from bag A and white ball from bag B.

c. A blue ball from bag A and a white ball from bag B.
d. A white ball from bag A and a blue ball from bag B.

Solutions:

In bag A, there are 8 balls, 5 blue and 3 white.
In bag B, there are 12 balls, 4 blue and 8 white.

a. Probability of blue from A and blue from B $= \frac{5}{8} \times \frac{4}{12} = \frac{5}{24}$
b. Probability of white from A and white from B $= \frac{3}{8} \times \frac{8}{12} = \frac{1}{4}$
c. Probability of blue from A and white from B $= \frac{5}{8} \times \frac{8}{12} = \frac{5}{12}$
d. Probability of white from A and blue from B $\frac{3}{8} \times \frac{4}{12} = \frac{1}{8}$

5. A card is taken at random from each of two ordinary packs of cards, pack A and pack B. What is the probability of getting:
   a. A red card from pack A and a red card from pack B.
   b. A diamond from pack A and a club from pack B.
   c. A king from pack A and a picture card (king, queen, jack) from pack B.
   d. A 10 from pack A and a 10 of clubs from pack B.
   e. An ace of hearts from each pack.

Solutions:

Note that half of the cards in an ordinary pack are red, and half are black. There are 52 cards in a pack, and 4 suits: diamonds, hearts, spades, and clubs. Thus, there are 4 cards of each number or picture card in the deck (4 sevens, 4 kings, and so on.) Use this information to find the probabilities.

a. Note that $P(\text{red}) = \frac{26}{52} = \frac{1}{2}$

   $P(r\text{ed from } A)$ and P (red from $B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

b. Note that $P(\text{diamond}) = P(\text{club}) = \frac{13}{52} = \frac{1}{4}$

   $P(\text{diamond from } A)$ and P (club from $B) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$

c. Note that $P(\text{king}) = \frac{4}{52} = \frac{1}{13}$ and $P(\text{picture card}) = \frac{12}{52} = \frac{3}{13}$

   $P(\text{king from } A)$ and P (picture card from $B) = \frac{1}{13} \times \frac{3}{13} = \frac{3}{169}$

d. Note that $P(10) = \frac{4}{52} = \frac{1}{13}$ and $P(10 \text{ of clubs}) = \frac{1}{52}$

   $P(10 \text{ from } A)$ and P (10 of clubs from $B) = \frac{1}{13} \times \frac{1}{52} = \frac{1}{676}$

e. Note that $P(\text{ace of hearts}) = \frac{1}{52}$

   $P(\text{ace of hearts from } A)$ and P (ace of hearts from $B) = \frac{1}{52} \times \frac{1}{52} = \frac{1}{2{,}704}$

Practice

1. The probability that it will rain tomorrow is $\frac{2}{3}$. The probability that Yasmin will forget her umbrella is $\frac{3}{4}$. What is the probability that it will rain tomorrow and Yasmin will forget her umbrella?
2. A bag contains 6 red balls and 5 black balls. A ball is picked from the bag, replaced and a second ball picked. Find the probability that:
   a. The first is black and the second is red.
   b. Both are black.
   c. Both are red.
3. The probability that a man travels by motor bike is 0.4 and the probability that he is late for his programme is 0.7. Find the probability that the man:
   a. Was late for his programme and travelled by motor bike
   b. Was late for his programme and did not travel by motor bike
   c. Was not late for his programme and travelled by motor bike
   d. Was not late for his programme and did not travel by motor bike
4. A die is thrown twice. What is the probability that:
   a. Two odd numbers are obtained
   b. The first throw is an odd number and the second throw is a number greater than 2
5. In a primary school, 80% of the boys and 65% of the girls walk to school. If a boy and a girl are chosen at random, what is the probability that:
   a. Both of them walk to school
   b. Neither of them walk to school
6. A bag contains 8 blue balls, 4 red balls and 6 white balls. A ball is drawn from the bag without replacement and a second ball drawn. What is the probability that:
   a. They are both blue
   b. They are both red
   c. They are both white
   d. The first is white and the second is blue
   e. The first is white and the second is red
   f. The first is blue and the second is red
7. A ball is drawn at random from each of two bags A and B. Bag A contains 7 white balls and 3 red balls, bag B contains 4 white balls and 5 red balls. What is the probability of getting:
   a. A white ball from bag A and a white ball from bag B
   b. A red ball from A and a red ball from B
   c. A white from A and a red from B
   d. A red from A and a white from B

Lesson Title: Illustration of probabilities	**Theme:** Probability and Statistics
Practice Activity: PHM4-L087	**Class:** SSS 4

Learning Outcome

By the end of the lesson, you will be able to use outcome tables and tree diagrams to illustrate probability and solve problems.

Overview

This lesson is on creating and using illustrations to solve probability problems. This lesson covers two methods of illustrating probabilities: outcome tables and tree diagrams. Venn diagrams are illustrated in the next lesson.

Outcome Tables

When dealing with the probability of an event occurring, it is very important to identify all the outcomes of the experiment. One way in which this can be done for outcomes which are all equally likely is by systematically listing all of them in a sample space as shown for throwing a die. That is $S = \{1, 2, 3, 4, 5, 6\}$

However, when we have to identify the outcomes for two equally likely events occurring, listing can result in missing out some of the outcomes.

Instead, we use an outcome or 2-way table to identify all the outcomes. Drawing a table means we do not have to calculate the required probabilities. We can just count them off the table.

The 2-way table in Figure 1 shows all the outcomes when tossing 2 fair coins.

The 2-way table in Figure 2 shows all the outcomes for rolling 2 unbiased dice.

Second coin

First coin		H	T
	H	HH	HT
	T	TH	TT

Figure 1

Second die

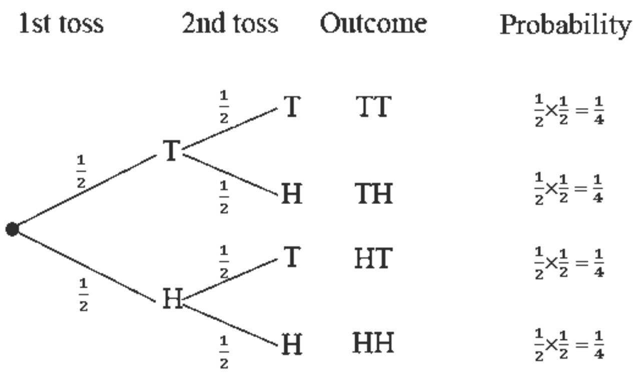
First die		1	2	3	4	5	6
	1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
	2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
	3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
	4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
	5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
	6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Figure 2

Tree Diagrams

Outcome tables cannot be used when the events are not equally likely to occur or when we have more than 2 events. In such situations, we use a tree diagram where every branch represents an event together with its probability of occurring.

This is an example of a tree diagram that shows 2 coin tosses.



Solved Examples

1. Two fair coin are tossed. Using the table in Figure 1, find the probability that:
   a. Both coins show heads
   b. Only one coin shows a tail
   c. Both coins land the same way up

   Solutions:

   Calculate the required probabilities.

   From the table: $n(S) = 4$

   a. $P(\text{both coins show head}) = \frac{1}{4}$

   b. $P(\text{only one coin show a tail}) = \frac{2}{4} = \frac{1}{2}$

   c. $P(\text{both coins land the same way up}) = \frac{2}{4} = \frac{1}{2}$

2. Two unbiased dice are thrown. Using the table In Figure 2, find the probability that:
   a. Both coins show an even number
   b. At least one coin shows a 5
   c. No coin shows a 5
   d. Both coins show the same number

   Solutions:

   Given: Figure 2, find the required probabilities

   $n(S) = 36$

   a. $P(\text{both coins show even number}) = \frac{9}{36} = \frac{1}{4}$

   b. $P(\text{at least one coin shows 5}) = \frac{11}{36}$

   c. $P(\text{no coin shows 5}) = 1 - P(\text{at least one coin shows 5})$

$$= 1 - \frac{11}{36} = \frac{25}{36}$$

d. $P(\text{both coins show the same numbe} = \frac{6}{36} = \frac{1}{6}$

3. A fair coin is tossed twice. What is the probability of getting:

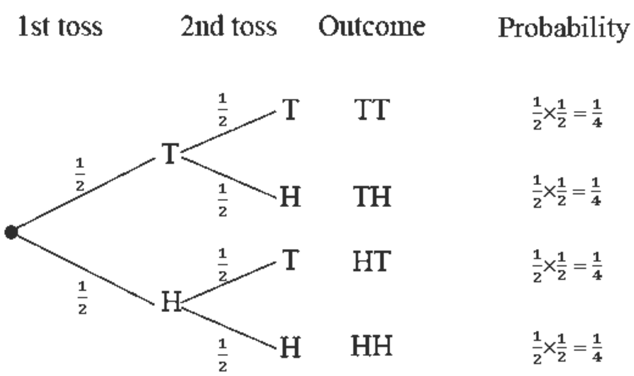
   a. Two heads b. No heads c. Only one head

   Solutions:

   Step 1. Draw the tree diagram showing all the outcomes.

   Step 2. Find the probability of each outcome.

   The probability of each outcome is obtained by multiplying the probabilities of the branches leading to that outcome.



   Step 3. Find and write the required probabilities.

   a. The probability of 2 heads is given by the bottom branch and is $\frac{1}{4}$.

   b. The probability of no heads is given by the top branch and is $\frac{1}{4}$.

   c. The probability of only head is given by the 2 middle branches –

   The combined probability is: $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

   The probabilities should all add up to 1.

   $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$ as expected

4. A die has 6 faces of which 3 are black, 2 white and 1 yellow. If the die is rolled twice, what is the probability of getting:

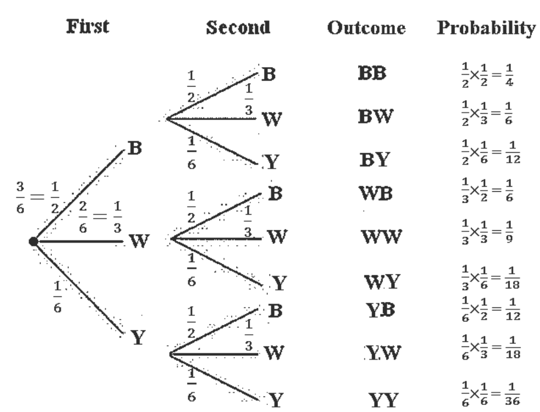
   a. Both faces are yellow

   b. Both faces the same colour

   c. Neither face is black

   Solutions:

   Draw the tree diagram →



   $P(\text{both yellow}) = P(YY) = \frac{1}{36}$

   $P(\text{same colour}) = P(BB) + P(WW) + P(YY)$

$$= \frac{1}{4} + \frac{1}{9} + \frac{1}{36} = \frac{7}{18}$$

$$P(\text{neither is black}) = P(WW) + P(WY) + P(YW) + P(YY)$$

$$= \frac{1}{9} + \frac{1}{18} + \frac{1}{18} + \frac{1}{36} = \frac{1}{4}$$

Practice

1. Two fair 4-sided spinners are spun and the difference between the numbers is calculated.
   a. Copy and complete the outcome table showing all the possible outcomes.

Spinner A

Spinner B

1	2	3	4	
1	0	1		
2	1			
3				
4				

   What is the probability of getting a difference of:
   b. 0 c. 3
   d. 4 e. 1 or 2
2. Ibrahim has two packets of coloured pencils. In the first packet, there is a pink pencil, a blue pencil, a yellow pencil and a white pencil. In the second packet, there is a pink, a yellow and a blue pencil. Ibrahim takes a pencil at random from each packet.
   Draw an outcome table to show all the possible pairs of colours.
   Find the probability that the pencils will be:
   a. Both yellow b. The same colour c. Different colours
3. A school cook decides at random which of 3 sauces she will cook each day. He chooses from cassava leaves (C), potato leaves (P) and groundnut soup (G).
   a. Draw an outcome table showing all the possible outcomes from two consecutive days.
   What is the probability of:
   b. Cassava leaves on both days c. The same sauce on both days
   d. Potato leaves or groundnut soup on both days
   e. Kadija does not like groundnut soup. What is the probability she will not eat lunch in school for two consecutive days?
4. A bag contains 5 red balls and 3 green balls. A ball is chosen from the bag and then replaced. A second ball is chosen.
   a. Show the possible ways of selecting the balls on a tree diagram.
   What is the probability that:
   b. They are both red c. One is red and one is green
   d. At least one is red e. At most one is red
5. There are 10 pencils in a case. Three of the pencils are HB pencils. A pencil is taken at random from the pencil case and returned. A second pencil is now taken from the pencil case and then returned.
   a. Draw a tree diagram to show all the possible outcomes.
   b. What is the probability that only one of the pencils will be an HB pencil?

6. Victoria spins two spinners, A and B. The probability of getting a 6 on spinner A is 0.3. The probability of getting a six on spinner B is 0.45.
   a. Draw a tree diagram to show all the possible outcomes.
   Workout the probability of getting a 6 on:
   b. Neither spinner
   c. Only one spinner
   d. Spinner B only

Lesson Title: Probability problem solving	**Theme:** Probability and Statistics
Practice Activity: PHM4-L088	**Class:** SSS 4

Learning Outcome
By the end of the lesson, you will be able to solve problems related to probability.

Overview

Problems for this lesson will cover the concepts from previous lessons on probability.

Solved Examples

1. A letter is selected at random from the letters of the English alphabet. What is the probability that the letter selected is from the words SIERRA LEONE?
   Solution:
   Identify the number of letters that appear in "Sierra Leone". There are 8 letters: s, i, e, r, a, l, o, n. Some letters appear more than once, but are only counted once. The probability is: $\frac{8}{26} = \frac{4}{13}$.

2. The probabilities that Sia, Hawa, and Foday will score a goal are $\frac{3}{5}$, $\frac{2}{3}$, and $\frac{1}{4}$, respectively. Find the probability that only Sia will score a goal.
   Solution:
   If only Sia scores a goal, then Hawa and Foday will not score a goal. Subtract from 1 to find the probability that each will not score a goal:
   $$\text{Pr(Hawa will not score)} = 1 - \frac{2}{3} = \frac{1}{3}.$$
   $$\text{Pr(Foday will not score)} = 1 - \frac{1}{4} = \frac{3}{4}$$
   Multiply the probability that Sia will score by the probabilities that Hawa and Foday will not score:
   $$\text{Pr(only Sia will score)} = \frac{3}{5} \times \frac{1}{3} \times \frac{3}{4} = \frac{9}{60} = \frac{3}{20}$$

3. Mohamed applied to enroll at 2 universities, A and B. The probability that he will be accepted to university A is 0.6. The probability that he will **not** be accepted to university B is 0.3. What is the probability that he will:
   a. Be accepted to both universities
   b. Be accepted to exactly 1 university
   c. Be accepted to neither university
   Solutions:

a. First, find the probability that he will be accepted to university B:

   Pr(Accepted to B) $= 1 - 0.3 = 0.7$

   Multiply the probability that he will be accepted to university A by the probability that he will be accepted to university B.

   Pr(Accepted to A and B) $= 0.6 \times 0.7 = 0.42$

b. Find the probability that he will be accepted to only A and not B by subtracting the probability that he will be accepted to both by the probability that he will be accepted to A:

   Pr(Accepted to only A) $= 0.6 - 0.42 = 0.18$

   Follow the same process for B:

   Pr(Accepted to only B) $= 0.7 - 0.42 = 0.28$

   Add to find the probability that he will be accepted to only A or only B:

   Pr(Accepted to only A or B) $= 0.18 + 0.28 = 0.46$

c. Multiply the probability that he will not be accepted to university A by the probability that he will not be accepted to university B:

   Pr(Accepted to neither A nor B) $= 0.4 \times 0.3 = 0.12$

4. Fatu has 30 oranges in a box. Some are ripe while others are not. If the probability of selecting a ripe orange is $\frac{1}{5}$, calculate the number of:

   a. Unripe oranges
   b. Ripe oranges which should be added to the box such that the probability of picking a ripe orange will be $\frac{1}{2}$.

   Solutions:

   a. Note that the total number of oranges is 30, and the probability of selecting an unripe orange is $1 - \frac{1}{5} = \frac{4}{5}$. We also know that:

      Pr(choosing an unripe orange) $= \frac{\text{No.of unripe oranges}}{\text{Total no.of oranges}}$

      Therefore, we have $\frac{\text{No.of unripe oranges}}{\text{Total no.of oranges}} = \frac{\mathrm{U}}{30} = \frac{4}{5}$

      Cross multiply and solve for the number of unripe oranges (U):

$$\begin{aligned} \frac{\mathrm{U}}{30} &= \frac{4}{5} \\ 5U &= 30 \times 4 \\ U &= \frac{120}{5} = 24 \end{aligned}$$

      Answer: There are 24 unripe oranges.

   b. The probability of selecting a ripe orange is $\frac{1}{5}$, so the number of ripe oranges currently is $\frac{1}{5} \times 30 = 6$. A certain number of ripe oranges should be added to the total to create a probability of $\frac{1}{2}$. Let's call that number r. Then we have: $\frac{1}{2} = \frac{6+r}{30+r}$. This is based on the new probability, and adding r to both the total number, and the number of ripe oranges.

$$\begin{aligned} \frac{1}{2} &= \frac{6+r}{30+r} \\ 30+r &= 2(6+r) && \text{Cross multiply} \\ 30+r &= 12+2r \\ 30-12 &= 2r-r \\ 18 &= r \end{aligned}$$

Answer: 18 ripe oranges should be added.

Practice

Use an appropriate diagram to answer the question where necessary.

1. The table below shows the number of pupils with their body weights.

Weights(kg)	40	30	25	20	15	12
No. of pupils	11	9	3	6	6	5

   a. How many pupils are there?
   What is the probability that if a pupil is chosen at random the body weight is:
   b. 25 kg and above   c. Less than 20 kg   d. More than 30 kg
2. One hundred tickets are sold in a raffle to win a car.
   a. Ali buys one ticket. What is the probability that he wins the car?
   b. Eku buys five tickets. What is the probability that she wins the car?
3. The probability of an event X is $\frac{2}{3}$ while that of another event Y is $\frac{1}{7}$. If the probability of both X and Y is $\frac{5}{11}$, what is the probability of:
   a. Either X or Y   b. Neither X nor Y
4. Two unbiased dice are rolled at the same time. The scores are then multiplied together to get a score. What is the probability of getting:
   a. A score of 12   b. A score of more than 20
   c. A score of less than 8
5. Nine slips of paper are numbered 1 to 9. A slip is drawn at random. This is replaced before a second slip is drawn. What is the probability that one is an odd number and the other an even number?
6. A bag contains identical stones of which 12 are painted red, 16 are painted white and 8 are painted blue. Three stones are drawn from the bag one after the other without replacement. Find the probability that:
   a. Three are red.   b. The first is blue and the other two are red.
   c. They are of the same colour.

7. A bag contains papers with the names of 60 pupils in a class. If a paper is chosen at random, the probability of selecting a female pupil is $\frac{3}{5}$. How many males are in the class?
8. A letter is chosen at random from the alphabet. Find the probability that it is in either the word MATHS or SCIENCE.
9. A number has 3 digits formed by arranging 3, 4, and 5 randomly. Find the probability that the number is divisible by:
    a. Two
    b. Five
    c. Ten

Lesson Title: Mock Examination: Paper 1 – Multiple Choice	**Theme:** WASSCE Exam Preparation
Practice Activity: PHM4-L089	**Class:** SSS 4

Learning Outcomes
By the end of the lesson, you will be able to:
1. Complete a section of a mock WASSCE paper.
2. Answer multiple choice questions on various topics.

Overview

The WASSCE exam has 3 sections. Today, you will practise section 1, which consists of multiply choice questions.
Paper 1 – Multiple Choice

- Paper 1 is 1.5 hours, and consists of 50 multiple choice questions. It is worth 50 marks.
- This gives 1.8 minutes per problem, so time must be planned accordingly.
- The questions are drawn from all topics on the WASSCE syllabus.

The following is general advice for taking the WASSCE exam:

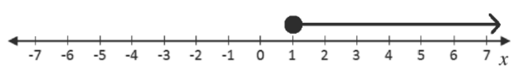
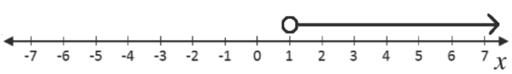
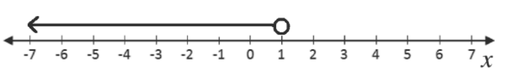
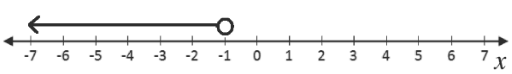
- Read and understand the instructions that you are given on the day of the examination.
- Always follow given instructions. For example, if you are asked to find a value using tables, you will lose marks if you use a calculator.
- Plan your time, and do not spend too much time on each question. If you do not understand a question, save it for last and move on to solve a question you understand.
- Check your answers to make sure that they are sensible in terms of the given question.
- Show all of your work in the body of the question you are answering. Examiners may give some credit marks for rough work, but it must be on the examination paper.

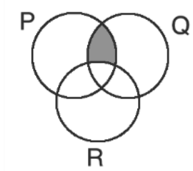
This is the first of 8 mock examinations. To be effective practice, each mock exam should be taken under exam conditions. This means that you should work independently, without referring to notes or books. If you have a calculator and/or geometry set, bring these to the mock exam. You should try to complete the test within the stated time. After completing the exam, check your answers in the Answer Key. If you miss a question, it is a good idea to study the related topic.

Practice
Complete the following 18 problems. You have 34 minutes. **Do not look at the answer key during the exam.**

1. Find the 7th term of the sequence: 4, 12, 36, …
   A. 22
   B. 60
   C. 2,916
   D. 8,748
2. Bintu draws the graphs of $y = x^2 + 2x - 3$ and $y = 3x - 1$ on the same axes. Which of these equations is she solving?
   A. $x^2 + 5x - 4 = 0$
   B. $x^2 - x - 2 = 0$
   C. $x^2 - x - 4 = 0$
   D. $x^2 - 5x + 2 = 0$
3. The population of pupils in a school is 625, of which 300 are girls. If this is represented on a pie chart, calculate the sectoral angle for girl pupils.
   A. $46°$
   B. $173°$
   C. $180°$
   D. $187°$
4. The table below shows the distribution of the scores of some pupils on a test. Calculate the mean score.

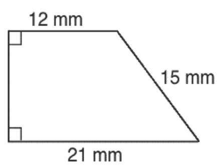
Scores	1-5	6-10	11-15	16-20
Frequency	1	2	5	2

   A. 10
   B. 11
   C. 12
   D. 13
5. Illustrate graphically the solution of $\frac{2x}{3} - \frac{5}{6} > -\frac{x}{6}$.
   A. 
   B. 
   C. 
   D. 
6. Make a the subject of the relation: $b = \sqrt{\frac{2a+5}{a-1}}$.
   A. $a = \frac{b^2+5}{b^2-2}$
   B. $a = \frac{b^2-2}{b^2+5}$
   C. $a = \frac{b+5}{b-2}$
   D. $a = \frac{b-2}{b+5}$
7. Describe the shaded portion in the diagram.
   A. $R \cap (P \cap Q)'$
   B. $R \cup (P \cap Q)'$
   C. $R' \cup (P \cap$
   D. $R' \cap (P \cap Q)$

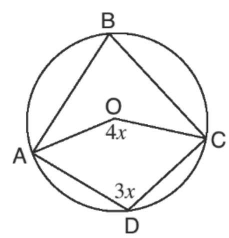


8. In a circle of radius r, a chord 24 cm long is 16 cm from the centre of the circle. Find the value of r, to the nearest cm.
   A. 16 cm
   B. 20 cm
   C. 29 cm
   D. 40 cm
9. Half of a number added to 3 times that number gives 77. Find the missing number.
   A. 7
   B. 11
   C. 22
   D. 38.5
10. A woman's eye level is 1.8 m above the horizontal ground and 12 m from a flag pole. If the pole is 5.4 m tall, calculate the angle of elevation of the top of the pole from her eyes. Give your answer to the nearest degree.
    A. 17°
    B. 24°
    C. 66°
    D. 73°

11. How many sides has a regular polygon with interior angles of 135°?
    A. 5
    B. 6
    C. 7
    D. 8
12. A bag of rice can feed 20 people for 12 days. How many days will it last for 80 people?
    A. 3 days
    B. 6 days
    C. 12 days
    D. 24 days
13. Simplify: $\frac{a}{2a+4b} + \frac{b}{a+2b} - \frac{1}{2}$
    A. 0
    B. $\frac{1}{2}$
    C. 1
    D. $\frac{2b}{a+2b}$
14. Calculate the area of the trapezium to the nearest square millimetre.
    A. 198 mm^2
    B. 248 mm^2
    C. 396 mm^2
    D. 495 mm^2



15. Simplify: $36^{\frac{1}{2}} \times 8^{-\frac{2}{3}}$
    A. $\frac{3}{4}$
    B. $1\frac{1}{2}$
    C. $4\frac{1}{2}$
    D. 24
16. The volume of a cuboid is 162 m^3. If the length, width, and height are in the ratio 2 : 1 : 3 respectively, find its total surface area.
    A. 36 m^2
    B. 99 m^2
    C. 198 m^2
    D. 324 m^2
17. Sia spent $\frac{1}{2}$ of her money on food, $\frac{1}{5}$ on school supplies and saved the rest. If she saved ₦9,000.00, how much did she spend on food?
    A. ₦2,700.00
    B. ₦9,000.00
    C. ₦15,000.00
    D. ₦ 30,000.00
18. The diagram is a circle with centre O. ABCD are points on the circle. Find the value of $\angle ABC$.



    A. 36°
    B. 72°
    C. 108°
    D. 144°

Lesson Title: Mock Examination: Paper 1 – Multiple Choice	**Theme:** WASSCE Exam Preparation
Practice Activity: PHM4-L090	**Class:** SSS 4

Learning Outcomes
By the end of the lesson, you will be able to:
1. Complete a section of a mock WASSCE paper.
2. Answer multiple choice questions on various topics.

Overview

The WASSCE exam has 3 sections. Today, you will practise section 1, which consists of multiply choice questions. This is the second of 2 mock exams on section 1.

What are some important things to keep in mind when taking an exam? Share ideas with your classmates. Here are some ideas:

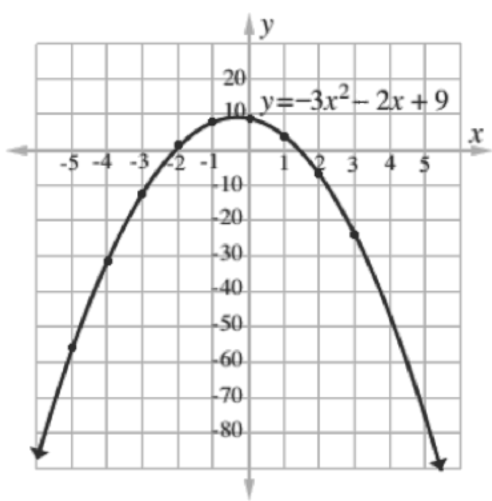
- Plan your time. Do not spend too much time on one problem.
- If you don't know a multiple-choice answer or run out of time to solve a problem, make an educated guess. Eliminate any answers that seem impossible, and choose an answer that looks sensible.
- If you complete the exam, take time to check your solutions. If you notice an incorrect answer, double check it before changing it.

Practice

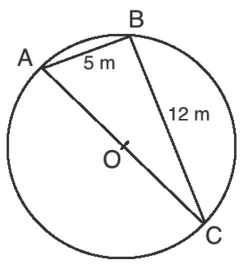
Complete the following 18 problems. You have 34 minutes. **Do not look at the answer key during the exam.**

1. If $U = \{t, u, v, w, x, y, z\}$, $A = \{t, v, x, z\}$ and $B = \{x, y, z\}$ find $(A \cap B)'$.
   A. $\{x, z\}$
   B. $\{t, v, y\}$
   C. $\{t, v, x, y, z\}$
   D. $\{t, u, v, w, y\}$
2. x varies directly as y and inversely as z. If $x = 2$ when $y = 8$ and $z = 12$, Find x in terms of y and z.
   A. $x = \frac{3y}{z}$
   B. $x = \frac{3z}{y}$
   C. $x = \frac{4z}{3y}$
   D. $x = \frac{yz}{48}$

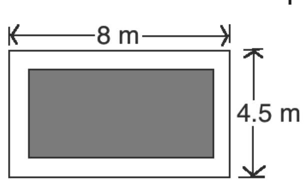
Use the graph of $y = -3x^2 - 2x + 9$ to answer questions 3 and 4.



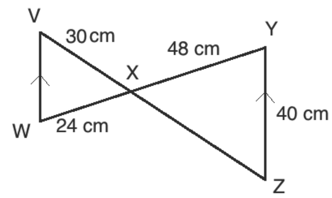
3. Which of the following is approximately equal to the smaller root of the equation?
   A. -3.0
   B. -2.1
   C. -0.2
   D. 1.5
4. Estimate the gradient at the point $x = 2$.
   A. -1.4
   B. -5
   C. -14
   D. 14
5. In the diagram, AC is a diameter of the circle with centre O. Find the radius of the circle.



   A. 5 m
   B. 6.5 m
   C. 7 m
   D. 13 m
6. The diagram shows a carpet laid in a room that is 8 metres long by 4.5 metres wide. There is a 0.75-metre margin between each wall and the carpet. Find the area of the carpet, correct to 1 decimal place.

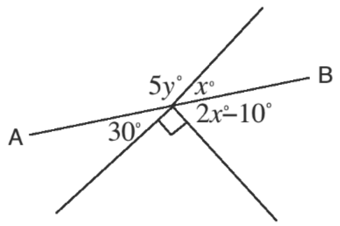


   A. 19.5 m^2
   B. 27.2 m^2
   C. 36.0 m^2
   D. 57.0 m^2
7. In the diagram, $VW||YZ$, $|VX| = 30$ cm, $|WX| = 24$ cm, $|XY| = 48$ cm, and $|YZ| = 40$ cm. Calculate $|VW|$.



   A. 15 cm
   B. 20 cm
   C. 25 cm
   D. 30 cm
8. Find the equation whose roots are -3 and $\frac{1}{2}$.
   A. $x^2 - 5x - 3 = 0$
   B. $2x^2 - 7x - 3 = 0$
   C. $2x^2 + 5x + 3 = 0$
   D. $2x^2 + 5x - 3 = 0$
9. Factorise completely: $2ax - 21by - 3bx + 14ay$.
   A. $3a(2x - 3y) + 4b(2x - 3y)$
   B. $2x(3a + 4b) - 3y(3a + 4b)$
   C. $2ax - 3b(7b + x) + 14ay$
   D. $(x + 7y)(2a - 3b)$
10. If $a = 4$, $b = 3$ and $x = -2$, evaluate $\frac{a}{b} + \frac{3x}{a} - 1\frac{1}{2}$.
   A. $-1\frac{2}{3}$
   B. $-1\frac{1}{3}$
   C. $-\frac{2}{3}$
   D. $-\frac{1}{3}$
11. A sector of a circle with radius 8 cm subtends an angle of 90° at the centre. Calculate its perimeter in terms of π.
   A. 4π
   B. $4(4 + \pi)$
   C. $4(2 + \pi)$
   D. $4(1 + \pi)$

12. If $\frac{8^x \times 2^{x+1}}{4^{3x}} = 1$, find the value of x.
    A. -1
    B. $-\frac{1}{2}$
    C. 0
    D. $\frac{1}{2}$
13. If $34_x = 10011_2$, find the value of x.
    A. 4
    B. 5
    C. 6
    D. 7
14. The area of a sector of a circle with radius 10 cm is 25π cm^2. If the sector is folded to form a cone, calculate the radius of the base of the cone.
    A. 2.5
    B. 2.5π
    C. 5
    D. 5π
15. Foday has 16 currency notes in his pocket, all of which are Le 1,000.00 or Le 5,000.00 notes. If he has a total of Le 52,000.00, how many Le 5,000.00 notes does he have?
    A. 7
    B. 8
    C. 9
    D. 10
16. In the diagram, AB is a straight line. Find the value of y.



    A. 29°
    B. 35°
    C. 36°
    D. 145°
17. A salesperson gave change of \$18.00 instead of \$22.00. Calculate his percentage error.
    A. 3.3%
    B. 18.2%
    C. 22.2%
    D. 81.8%
18. Fatu traveled 240 km from Bo to Freetown in 4 hours. What was her speed in m/s? Give your answer to 3 significant figures.
    A. 864 m/s
    B. 216 m/s
    C. 66.7 m/s
    D. 16.7 m/s

Lesson Title: Mock Examination: Paper 2A – Compulsory Questions	**Theme:** WASSCE Exam Preparation
Practice Activity: PHM4-L091	**Class:** SSS 4

Learning Outcomes
By the end of the lesson, you will be able to:
1. Complete a section of a mock WASSCE paper.
2. Answer essay questions on various topics.

Overview

The WASSCE exam has 3 sections. Today, you will practise section 2A, which consists of compulsory essay questions.

Paper 2 – Essay Questions

- Paper 2 consists of 13 essay questions in 2 sections – 2A and 2B.
- Paper 2 is worth 100 marks in total.
- Pupils will be required to answer 10 essay questions in all, across the 2 sections.
- This is an average of 15 minutes per question. However, keep in mind that section 2B is more complicated, therefore plan your time accordingly.

Paper 2A – Compulsory Questions

- Paper 2A is worth 40 marks.
- There are 5 **compulsory** essay questions in paper 2A.
- Compulsory questions often have multiple parts (a, b, c, …). The questions may not be related to each other. Each part of the question should be completed.
- The questions in paper 2A are simpler than those in 2B, generally requiring fewer steps.
- The questions on paper 2A are drawn from the common area of the WASSCE syllabus (they are not on topics that are specific to certain countries).

Practice

Complete the following 3 problems. You have 34 minutes. **Do not look at the answer key during the exam.**

1. a. a. Simplify: $\frac{1\frac{1}{2}+2\frac{1}{3}}{2\frac{1}{2}-3\frac{3}{4}\times\frac{2}{5}}$

   b. Given that $(\sqrt{2}-3\sqrt{5})(\sqrt{2}+\sqrt{5})=a+b\sqrt{10}$, find a and b.

2. a. The table shows the number of children of the families living in a certain community.

Children	0	1	2	3	4	5
Frequency	1	4	3	5	4	3

a. Find the: i. Mode ii. Mean iii. Third quartile

b. If a pie chart were drawn for the data, what would be the angle of the sector showing families with 4 children?

3. a. In a class of 50 pupils, 35 offered chemistry (C), 23 offered French (F) and 13 offered neither of the 2 subjects. i) Draw the Venn diagram to represent the information. ii) How many pupils offered both chemistry and French? Iii) What is $n(C \cup F)$?

b. If $a = \frac{2x}{x+1}$ and $b = \frac{x-1}{x+1}$, express $\frac{a+b}{a-b}$ in terms of x.

Lesson Title: Mock Examination: Paper 2A – Compulsory Questions	**Theme:** WASSCE Exam Preparation
Practice Activity: PHM4-L092	**Class:** SSS 4

Learning Outcomes
By the end of the lesson, you will be able to:
1. Complete a section of a mock WASSCE paper.
2. Answer essay questions on various topics.

Overview

The WASSCE exam has 3 sections. Today you will practise section 2A, which consists of compulsory essay questions. This is the second of 3 mock exams on section 2A.

What are some important things to keep in mind when taking an exam? Share ideas with your classmates. Here are some ideas:

- Plan your time. Do not spend too much time on one problem.
- Show all of your working on the exam paper. Examiners can give some credit for rough work. Do not cross out your work.
- If you complete the exam, take time to check your solutions. If you notice an incorrect answer, double check it before changing it.

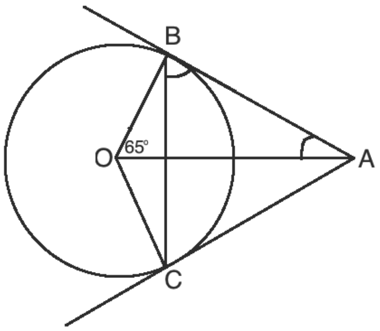
Practice

Complete the following 3 problems. You have 34 minutes. **Do not look at the answer key during the exam.**

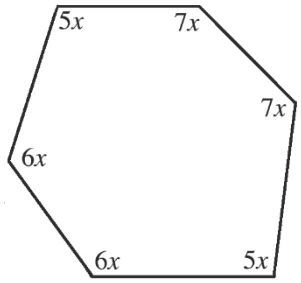
1. a. Make p the subject of the relation: $q = \sqrt{t^2p - \frac{rp}{t}}$

   b. If $2^{x+1} = 8^{3y}$ and $2x + y = 36$, find the value of $x - y$.
2. There are 45 pupils in a class. If the probability of selecting a female is $\frac{1}{3}$, calculate the number of:
   - Male pupils.
   - Female pupils who should be enrolled in the class such that the probability of picking a female pupil will be $\frac{1}{2}$.

3. In the given diagram, O is the centre of the circle. AB and AC are tangent lines, and $\angle AOB = 65°$. Calculate the measures of angles $\angle ABC$ and $\angle OAB$.



b. Find the value of x in the diagram below, in degrees.



Lesson Title: Mock Examination: Paper 2A – Compulsory Questions	Theme: WASSCE Exam Preparation
Practice Activity: PHM2-L093	**Class:** SSS 4

Learning Outcomes
By the end of the lesson, you will be able to:
1. Complete a section of a mock WASSCE paper.
2. Answer essay questions on various topics.

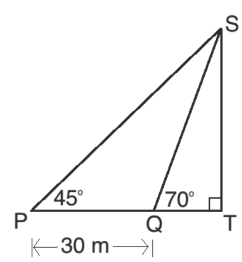
Overview

The WASSCE exam has 3 sections. Today, you will practise section 2A, which consists of compulsory essay questions. This is the third of 3 mock exams on section 2A.

Practice

Complete the following 3 problems. You have 34 minutes. **Do not look at the answer key during the exam.**

1. a. Simplify: $(3x+y)^2-(y-2x)^2$
   b. Given that $\sin x=\frac{4}{5}$ and $0° \leq x \leq 90°$, find $\frac{2\cos x-3\sin x}{\tan x}$.
2. Foday walked 3 kilometres from his house to school on a bearing of 45°. After school, he walked 4 kilometres to the market on a bearing of 135°. How far is he from his house?
3. In the diagram, two points P and Q are on the same horizontal as the base of a vertical pole, ST. P and Q are 30 metres from each other. Find, to 3 significant figures:
   - The height of the pole.
   - |PS|



Lesson Title: Mock Examination: Paper 2B – Advanced Questions	Theme: WASSCE Exam Preparation
Practice Activity: PHM4-L094	**Class:** SSS 4

Learning Outcomes

By the end of the lesson, you will be able to:

1. Complete a section of a mock WASSCE paper.
2. Select and solve advanced essay questions on various topics.

Overview

The WASSCE exam has 3 sections. Today, you will practise section 2B, which consists of advanced essay questions.

Paper 2 – Essay Questions

- Paper 2 consists of 13 essay questions in 2 sections – 2A and 2B.
- Paper 2 is worth 100 marks in total.
- Pupils will be required to answer 10 essay questions in all, across the 2 sections.
- This is an average of 15 minutes per question. However, keep in mind that section 2B is more complicated, and plan your time accordingly.

Paper 2B – Advanced Questions

- Paper 2B is worth 60 marks.
- There are 8 essay questions in paper 2A, and candidates are expected to answer 5 of them.
- Questions on section 2B have a greater length and difficulty that section 2A.
- A maximum of 2 questions (from among the 8) may be drawn from parts of the WASSCE syllabus that are not meant for Sierra Leone. These topics are not in the Sierra Leone national curriculum for secondary schools. Candidates from Sierra Leone may choose to answer such questions, but it is not required.
- Choose 5 questions on topics that you are more comfortable with.

Practice

Complete any 2 of the following 4 problems. You have 34 minutes. **Do not look at the answer key during the exam.**

1. Y is 80 km away from X on a bearing of 120°. Z is 100 km away from X on a bearing of 225°. Find, correct to 3 significant figures: a. The distance of Z from Y; b. The bearing of Z from Y.

2. a. Copy and complete the table of values for $y = 5\cos x + 2\sin x$ to one decimal place.

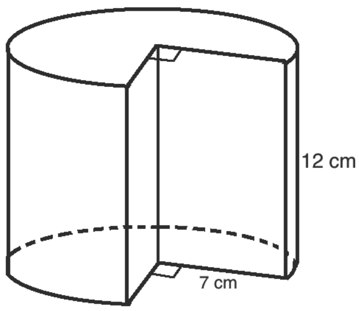
x	$0°$	$30°$	$60°$	$90°$	$120°$	$150°$	$180°$	$210°$
y	5.0		4.2		-0.8			-5.3

b. Using a scale of 2 cm to $30°$ on the x-axis and 2 cm to 1 unit on the y-axis, draw the graph of $5\cos x + 2\sin x = 0$ for $0° \leq x \leq 210°$.

c. Use your graph to solve the equation $5\cos x + 2\sin x = 0$, correct to the nearest degree.

d. Find the maximum value of y, correct to 1 decimal place.

3. The solid given below is a cylinder with a segment of 90° removed. Calculate the:
a. Volume of the solid; b. Surface area of the solid. Use $\pi = \frac{22}{7}$.



4. Using a ruler and a pair of compasses only, construct: a. Triangle ABC in which $|AB| = 6$ cm, $|BC| = 5$ cm and $\angle ABC = 60°$. Measure AC;
b. In a. above, locate by construction a point D such that CD is parallel to AB and D is equidistant from points A and C. Measure $\angle BAD$.

Lesson Title: Mock Examination: Paper 2B – Advanced Questions	**Theme:** WASSCE Exam Preparation
Practice Activity: PHM2-L095	**Class:** SSS 4

Learning Outcomes
By the end of the lesson, you will be able to:
1. Complete a section of a mock WASSCE paper.
2. Select and solve advanced essay questions on various topics.

Overview

The WASSCE exam has 3 sections. Today, you will practise section 2B, which consists of advanced essay questions. This is the second of 3 mock exams on section 2B.

Remember that section 3B has 8 questions, and you must choose 5 of them to solve. It is a good idea to skim read the questions quickly before getting started. If you see a topic that you are familiar and comfortable with, work that problem.

On the day of the exam, remember to show all of your work on the test paper.

Practice

Complete any 2 of the following 4 problems. You have 34 minutes. **Do not look at the answer key during the exam.**

1. a. Use logarithm tables to evaluate $\frac{20.3\times\sqrt{1.568}}{2.34\times1.803}$.
   b. Mr. Bangura has 7 books on his shelf, 3 Mathematics books and 4 science books. Of these, he selects 2 at random, one after the other, with replacement. Find the probability that:
      i. Both were Mathematics books.
      ii. One was a Mathematics book and one was a science book.
2. The frequency distribution table shows the marks achieved by 100 pupils in a Mathematics test.

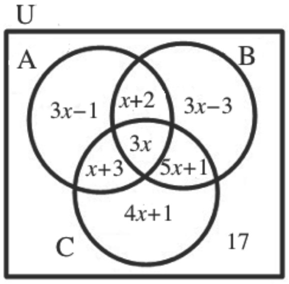
Marks (%)	1-10	11-20	21-30	31-40	41-50	51-60	61-70	71-80	81-90	91-100
Frequency	2	4	3	6	8	12	20	24	14	7

   a. Draw a cumulative frequency curve for the distribution.
   b. Use the graph to find the:
      i. 60^{th} percentile
      ii. Probability that a pupil passed the test if the pass mark was fixed at 55%.
3. The table is for the relation $y = px^2 + x + q$.

x	-4	-3	-2	-1	0	1	2	3
y		9			-6		4	15

a. i. Use the table to find the values of p and q.
   ii. Copy and complete the table.
b. Using scales of 2 cm to 1 unit on the x-axis, and 2 cm to 3 units on the y-axis, draw the graph of the relation for $-4 \leq x \leq 3$.
c. Use the graph to find:
   i. y when $x = 2.4$.
   ii. x when $y = -2$.

4. a. In the Venn diagram, A, B and C are subsets of the universal set U. If $n(U) = 120$, find:
   i. The value of x
   ii. $n(A \cup B \cup C')$



b. Given that $4\sin(x + 3.5) - 1 = 0$ and $0° \leq x \leq 90°$, calculate, correct to the nearest degree, the value of x.

Lesson Title: Mock Examination: Paper 2B – Advanced Questions	**Theme:** WASSCE Exam Preparation
Practice Activity: PHM2-L096	**Class:** SSS 4

Learning Outcomes
By the end of the lesson, you will be able to:
1. Complete a section of a mock WASSCE paper.
2. Select and solve advanced essay questions on various topics.

Overview

The WASSCE exam has 3 sections. Today, you will practise section 2B, which consists of advanced essay questions. This is the third of 3 mock exams on section 2B.

Practice

Complete any 2 of the following 4 problems. You have 34 minutes. **Do not look at the answer key during the exam.**

1. a. Copy and complete the following table for multiplication in modulo 11.

$\otimes$	1	2	4	6	8
1	1	2	4	6	8
2	2				
4	4				
6	6				
8	8				

Use the table to:

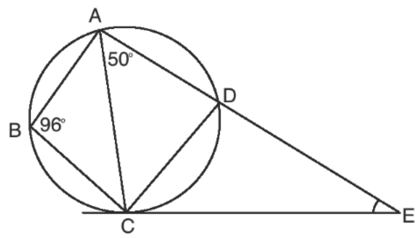
i. Evaluate $(8 \otimes 6) \otimes (4 \otimes 6)$.
ii. Find the truth set of $8 \otimes m = 4$.

b. When a fraction is simplified to its lowest term, it is equal to $\frac{2}{3}$. The numerator of the fraction when doubled is 12 greater than the denominator. Find the fraction.

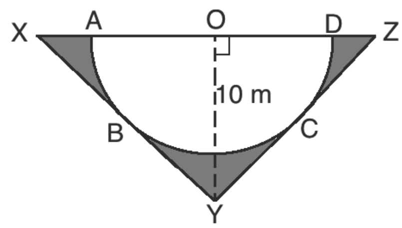
2. The table shows the distribution of outcomes when a die is thrown 50 times. Calculate the: a. Mean deviation; b. Probability that a score selected at random is at least 3.

Scores	1	2	3	4	5	6
Frequency (f)	8	7	10	11	5	9

3. a. In the diagram, CE is tangent to circle $ABCD$, $\angle ABC = 96°$ and $\angle CAD = 50°$. Find the measure of $\angle CED$.



   b. In the diagram, semi-circle $ABCD$ with centre O is inscribed in an isosceles triangle XYZ, where $|OY| = 10$ m and $\angle XYZ = 94°$. Find, correct to 3 significant figures: a. The area of semi-circle $ABCD$. b. The area of the shaded portion. (Take $\pi = \frac{22}{7}$)



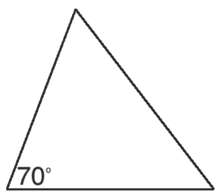
4. A school received $5,000.00 from a group of alumni to make improvements. A committee decided to spend 20% on new furniture, 30% on new books, 15% on teacher training, and 35% on scholarships for pupils.
   a. Represent this information on a pie chart.
   b. Calculate, correct to the nearest whole number, the percentage increase of the amount for scholarships over that for teacher training.

Answer Key – Term 2

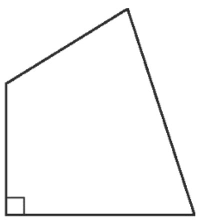
Lesson Title:	Measuring angles
Practice Activity:	PHM-L049

1. a. $\angle ABC = 28°$; b. $\angle RST = 163°$
2. Note that the following are example answers; your shapes may look different, but should have the given characteristics.

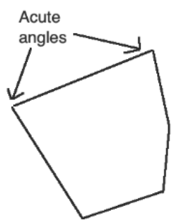
a.



b.



c.



3. a. Obtuse; b. Reflex; c. Acute; d. Reflex; e. Obtuse; f. Acute; g. Obtuse

Lesson Title:	Solving for angles – Part 1
Practice Activity:	PHM4-L050

1. $a = 140°$, $b = 40°$, $c = 40°$, $d = 140°$
2. $x = 61°$
3. $v = 48°$, $w = 82°$, $x = 42°$, $y = 48°$

Lesson Title:	Solving for angles – Part 2
Practice Activity:	PHM4-L051

1. a. $p = 65°$; b. $t = 75°$; c. $a = 30°$
2. a. $x = 30°$; b. $x = 36°$
3. $\angle BAC = 30°$
4. $\angle ACB = 88°$

Lesson Title:	Solving for angles – Part 3
Practice Activity:	PHM4-L052

1. a. $x = 14.4°$; b. $x = 5°$
2. 28 sides
3. 9 sides
4. a. $24°$; b. $156°$, c. $2{,}340°$
5. $x = 109°$

Lesson Title:	Solving for angles – Part 4
Practice Activity:	PHM4-L053

1. a. $\angle ABD = 35°$; b. $\angle ACB = 110°$; c. $\angle CAD = 75°$
2. a. $\angle OCD = 37°$; b. $\angle AOB = 112°$
3. a. $\angle AEB = 40°$; b. $\angle ABE = 100°$; c. $\angle DBC = 80°$; d. $\angle BDC = 29°$

Lesson Title:	Angle problem solving
Practice Activity:	PHM4-L054

1. a. $\angle BAD = 28°$; b. $\angle ADC = 28°$; c. $\angle ACD = 62°$
2. $a = 73°$
3. $a = 63°$; $b = 63°$; $c = 27°$

Lesson Title:	Conversion of units of measurement
Practice Activity:	PHM4-L055

1. a. $5{,}300$ m; b. 8,100 seconds; c. 9.5 l; d. 4,500 cm^3; e. 650 km; f. 76.5 t; g. 2,750,000 g.
2. 432 km/hr
3. $\text{Le}96{,}000{,}000.00$
4. 2.16×10^6 m

Lesson Title:	Area and perimeter of triangles and quadrilaterals
Practice Activity:	PHM4-L056

1. 156 cm^2
2. 24 cm^2
3. 140 cm^2
4. 108 cm^2

Lesson Title:	Trigonometric ratios
Practice Activity:	PHM4-L057

1. a. $46.47°$
   b. $50.23°$
   c. $34.32°$
2. a. $30°$
   b. $60°$
   c. $60°$
   d. $20.02°$
3. |BC|= 4.70 cm, |AC|= 1.71 cm, ∠B = $20°$
4. hypotenuse: 10 cm; angles: $53.13°$, $36.87°$
5. diagonal: 8.60 cm; angle: $35.54°$

Lesson Title:	Solving right-angled triangles
Practice Activity:	PHM4-L058

1. |AB| = 4 cm, |BC| = 6.93 cm, $\angle B = 30°$
2. |PQ| = 13.42 cm, $\angle P = 26.56°$, $\angle Q = 63.43°$
3. a. $\frac{2}{\sqrt{29}}$; b. $\frac{5}{\sqrt{29}}$; c. 1; d. $\frac{11}{18}$
4. a. $19.47°$; b. $4\sqrt{2}$ m or 5.66 m
5. 52.1 cm

Lesson Title:	Angles of elevation and depression
Practice Activity:	PHM4-L059

1. 60 m
2. $50.2°$
3. $57.3°$
4. a. 23.2 m; b. 25.8 m
5. a. 130.4 m; b. $72.9°$

Lesson Title:	The unit circle and trigonometric functions of larger angles
Practice Activity:	PHM4-L060

1. a. -0.8290; b.-1.280; c. -0.5; d. 2.747
2. a. $205.38°$, $336.42°$; b. 112.62, 292.62; c. $210°$, $330°$

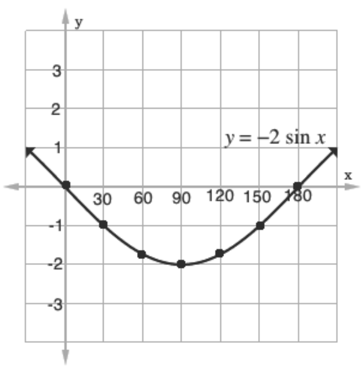
3. a. 0; b. $-\frac{\sqrt{3}}{6}$; c. $\frac{1}{4}+\sqrt{3}$

Lesson Title:	Graphs of trigonometric functions
Practice Activity:	PHM4-L061

1. Table of values:

x	$0°$	$30°$	$60°$	$90°$	$120°$	$150°$	$180°$
$-\sin x$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0
$-2\sin x$	0	-1	$-\sqrt{3}$	-2	$-\sqrt{3}$	-1	0

Graph:

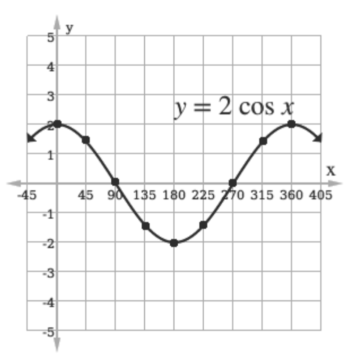


a. Estimate: $y=-0.3$
b. $x=30°, 150°$

2. Table of values:

x	$0°$	$45°$	$90°$	$135°$	$180°$	$225°$	$270°$	$315°$	$360°$
$\cos x$	1	$\frac{\sqrt{2}}{2}$	0	$-\frac{\sqrt{2}}{2}$	-1	$-\frac{\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1
$2\cos x$	2	$\sqrt{2}$	0	$-\sqrt{2}$	-2	$-\sqrt{2}$	0	$\sqrt{2}$	2

Graph:

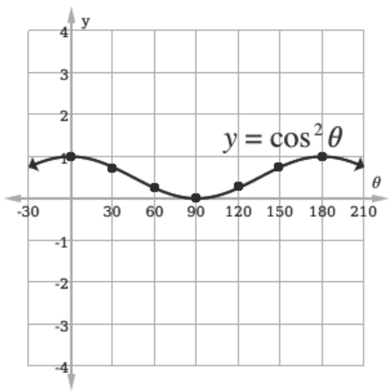


When $x = 200°$, y is approximately -1.8 or -1.9

3. Table of values:

θ	0°	30°	60°	90°	120°	150°	180°
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1
$\cos^2\theta$	1	$\frac{3}{4}$	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{3}{4}$	1

Graph:

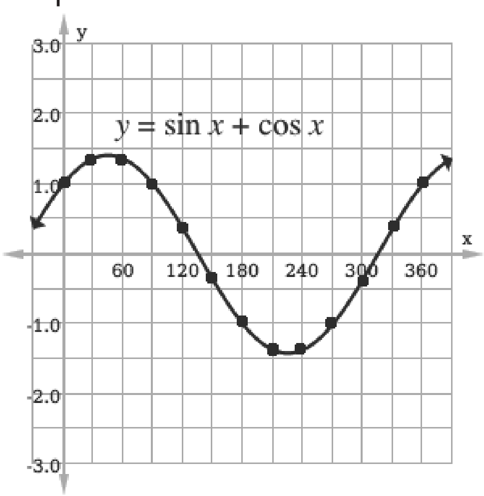


a. approximately 0.5; b. $\theta = 0°, 180°$

4. Table of values:

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$\sin x$	0	0.5	0.87	1	0.87	0.5	0	-0.5	-0.87	-1	-0.87	-0.5	0
$\cos x$	1	0.87	0.5	0	-0.5	-0.87	-1	-0.87	-0.5	0	0.5	0.87	1
$\sin x + \cos x$	1	1.37	1.37	1	0.37	-0.37	-1	-1.37	-1.37	-1	-0.37	0.37	1

Graph:

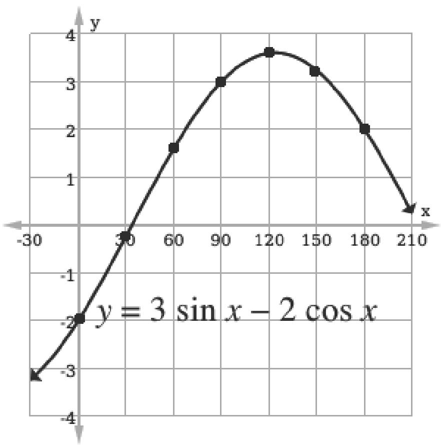


Answer: approximately $x = 105°, 345°$

5. Table of values:

x	$0°$	$30°$	$60°$	$90°$	$120°$	$150°$	$180°$
$3\sin x$	0	1.5	2.6	3	2.6	1.5	0
$2\cos x$	2	1.7	1	0	-1	-1.7	-2
$y = 3\sin x - 2\cos x$	-2	-0.2	1.6	3	3.6	3.2	2

Graph:



a. $x = 70°, 180°$; b. approximately $x = 35°$

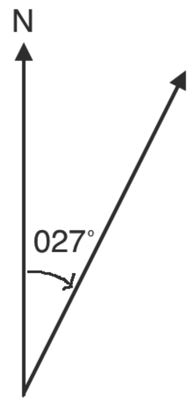
Lesson Title: Sine and Cosine Rules
Practice Activity: PHM2-L062

1. $17.7°$
2. $\angle B = 63.2°$ and $\angle C = 91.8°$
3. 34 mm
4. $34°$
5. $\angle B = 28.4°, \angle C = 88.6°, c = 10.2$ mm
6. $\angle P = 29.7°, \angle Q = 52.4°, \angle R = 97.9$

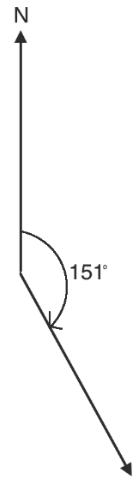
Lesson Title:	Three-figure bearings
Practice Activity:	PHM2-L063

1. Diagrams:

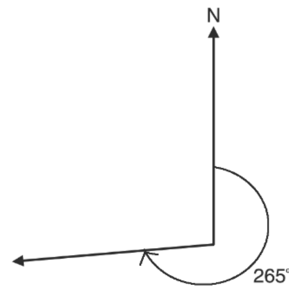
a.



b.



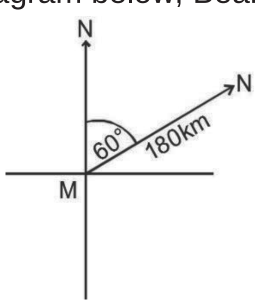
c.



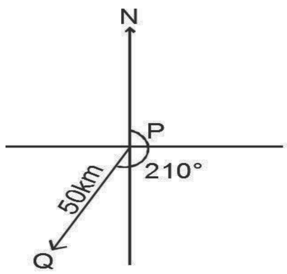
2. $020°$
3. 192°
4. a. $208°$; b. $028°$

Lesson Title:	Distance-bearing form
Practice Activity:	PHM2-L064

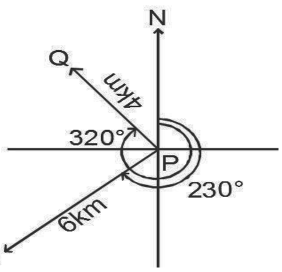
1. See diagram below; Bearing: $\overrightarrow{MN} = (180 \text{ km}, 060°)$



2. See diagram below; Bearing: $\overrightarrow{PQ} = (50 \text{ km}, 210°)$



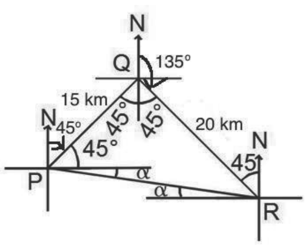
3. See diagram below; Bearing of Q: $\overrightarrow{PQ} = (4\text{ km}, 320°)$; Bearing of R: $\overrightarrow{PR} = (6\text{ km}, 230°)$



4. (7 km, $315°$)

Lesson Title:	Bearing problem solving – Part 1
Practice Activity:	PHM2-L065

1. a. 3.2 km b. $018°$
2. Distance= 5.4 km; Bearing= $338°$
3. Distance= 9.4 km; Bearing= $238°$
4. Distance= 64.0 km; Bearing= $141°$
5. See diagram below; b. i. $\overrightarrow{PR} = 25\text{ km}$; ii. $278°$



6. a. 13 km; b. $206°$

Lesson Title:	Bearing problem solving – Part 2
Practice Activity:	PHM2-L066

1. a. 100 km; b. $262°$
2. a. 49.3 cm.; b. $37.6°$
3. The bearing and distance from $P = (6.5 \text{ km}, 061°)$
4. a. 164 km; b. $108°$
5. 19 km
6. a. 13.09 km; b. $072°$

Lesson Title:	Circles
Practice Activity:	PHM2-L067

1. Completed table:

No.	Radius	Diameter	Circumference	Area
a.	12 m	24 m	75.43 m	452.57 m^2
b.	21 m	42 m	132 m	1,386 m^2
c.	98 mm	196 mm	616 mm	30,184 mm^2
d.	63 mm	126 mm	396 m	12,474 mm^2

2. 44 rose trees
3. 14 m
4. 113 cm^2
5. 25 cm
6. 173 cm^2
7. 44°
8. 11 cm
9. a. 4.72 cm; b. 15.0 cm

Lesson Title:	Circle Theorems 1 and 2
Practice Activity:	PHM2-L068

1. 17 cm
2. 10 cm
3. 0.5 cm
4. $8\sqrt{15}$ cm or 30.98 cm
5. a. $k = 90°$; b. $y = 48°$; c. $t = 90°$; $u = 45°$; $v = 45°$
6. a. $d = 106°$; $e = 254°$; $f = 127°$; b. $a = 75°$; $b = 210°$; $c = 105°$; c. $w = 98°$; $x = 131°$; $y = 49°$; $z = 131°$.

Lesson Title:	Circle Theorems 3, 4 and 5
Practice Activity:	PHM2-L069

1. a. $t = 101°$; $k = 114°$
   b. $q = 83°$; $r = 67°$; $s = 67°$; $t = 16°$; $u = 55°$ $v = 42°$
   c. $p = 30°$; $q = 30°$; $r = 30°$
   d. $c = 42°$; $d = 42°$; $e = 48°$
   e. $x = 45°$
   f. $a = 60°$

Lesson Title:	Circle Theorems 6 and 7
Practice Activity:	PHM2-L070

1. $|\mathrm{PA}| = |\mathrm{PB}| = 12\ \mathrm{cm}$
2. $\angle APB = 40°$; $\angle ABP = 70°$
3. a. $a = 90°$; $b = 23°$; $c = 67°$
   b. $a = 21°$; $b = 21°$; $c = 69°$
   c. $a = 40$; $b = 90°$; $c = 50°$
   d. $y = 48°$

Lesson Title:	Circle Theorem 8
Practice Activity:	PHM2-L071

1. $46°$
2. $60°$
3. $83°$
4. $35°$

Lesson Title:	Circle Problem Solving
Practice Activity:	PHM2-L072

1. $38°$
2. $54°$
3. $80°$
4. $55°$
5. a. $48°$; b. $48°$
6. $a = 80°$, $b = 50°$, $c = 63°$

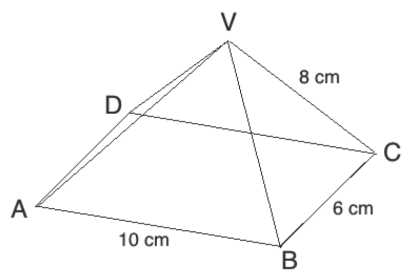
Lesson Title:	Surface Area
Practice Activity:	PHM4-L073

1. a. $616\ \mathrm{cm}^2$; b. $38.5\ \mathrm{m}^2$

2. a. 116 cm^2; b. 100 cm^2
3. 319 cm^2
4. 555.5 cm^2
5. 624 cm^2

Lesson Title:	Volume
Practice Activity:	PHM4-L074

1. Volume of oil $= 10{,}048 \text{ cm}^3$, height= 22.33 cm
2. 60 ball bearings
3. 214 m^3
4. a. Diagram:



a. i. 5.48 cm; ii. $43.2°$; iii. 109.60 cm^3

Lesson Title:	Operations on vectors
Practice Activity:	PHM4-L075

1. i. $\begin{pmatrix} 7 \\ -1 \end{pmatrix}$; ii. $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$; iii. $\begin{pmatrix} 7 \\ -8 \end{pmatrix}$
2. i. $\begin{pmatrix} 8 \\ 1 \end{pmatrix}$; ii. $\begin{pmatrix} -1 \\ -4 \end{pmatrix}$; iii. $\begin{pmatrix} 4 \\ -5 \end{pmatrix}$; iv. $\begin{pmatrix} 12 \\ 7 \end{pmatrix}$
3. i. $\begin{pmatrix} 11 \\ 11 \end{pmatrix}$; ii. $\begin{pmatrix} 7 \\ 3 \end{pmatrix}$; iii. $\begin{pmatrix} 12 \\ 4 \end{pmatrix}$; iv. $\begin{pmatrix} -12 \\ -4 \end{pmatrix}$
4. i. $\begin{pmatrix} -1 \\ -7 \end{pmatrix}$; ii. $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$; iii. $\begin{pmatrix} -7 \\ 4 \end{pmatrix}$
5. i. $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$; ii. $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$; iii. $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$; iv. $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$; v. $\begin{pmatrix} 1 \\ 11 \end{pmatrix}$
6. i. $\begin{pmatrix} 14 \\ -3 \end{pmatrix}$; ii. $\begin{pmatrix} -1 \\ -11 \end{pmatrix}$; iii. $\begin{pmatrix} -6 \\ 12 \end{pmatrix}$; iv. $\begin{pmatrix} 34 \\ 21 \end{pmatrix}$
7. i. $\begin{pmatrix} 1 \\ -5 \end{pmatrix}$; ii. $\begin{pmatrix} 11 \\ 2 \end{pmatrix}$; iii. $\begin{pmatrix} -17 \\ 4 \end{pmatrix}$

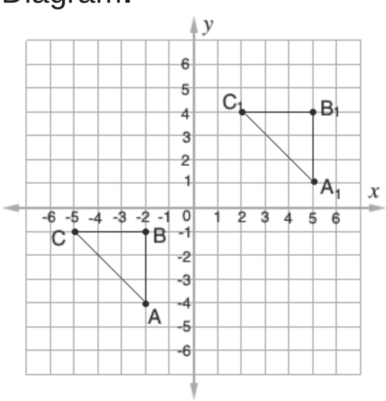
Lesson Title:	Magnitude and direction of vectors
Practice Activity:	PHM4-L076

1. a. 8.6; b. 4.5
2. a. $4\sqrt{10}$; b. 12.65
3. a. $\overrightarrow{XY} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$ $\overrightarrow{YZ} = \begin{pmatrix} -10 \\ 0 \end{pmatrix}$ $\overrightarrow{ZX} = \begin{pmatrix} 4 \\ -8 \end{pmatrix}$ b. ΔXYZ is Isosceles because $\left|\overrightarrow{XY}\right| = \left|\overrightarrow{YZ}\right| = 10$

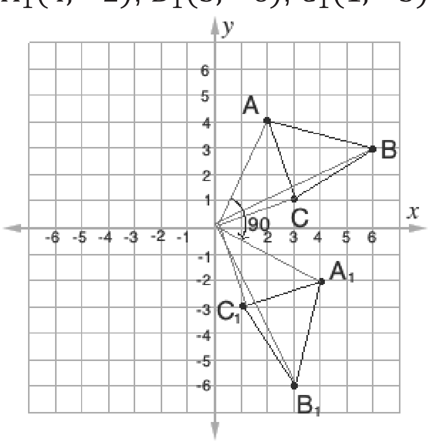
4. $y = 24$
5. $x = 5$
6. a.$108°$; b. $207°$
7. a. Q(2,-2); b. $323°$
8. a.$x = -1$ and $y = 3$; b. $\binom{1}{5}$; c. $\binom{-1}{-5}$
9. a. $\binom{-1}{6}$; b. $171°$

Lesson Title:	Transformation
Practice Activity:	PHM4-L077

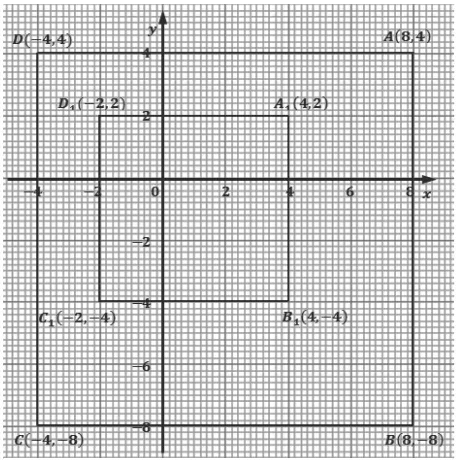
1. $D_1(1,-4)$, $E_1(3,-2)$, $F_1(4,-5)$
2. $\binom{-21}{39}$
3. Diagram:



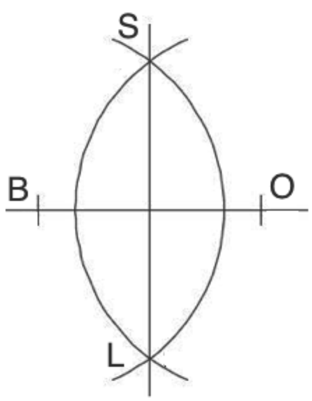
4. $A_1(4,-2)$, $B_1(3,-6)$, $C_1(1,-3)$; Diagram:



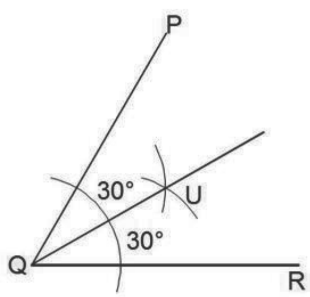
5. $A = (-4,5)$
6. a. $\mathbf{v} = \binom{3}{7}$; $Q = (-8,-9)$
7. $A_1(4,2), B_1(4,-4)$, $C_1(-2,-4)$, $D_1(-2,2)$; diagram:



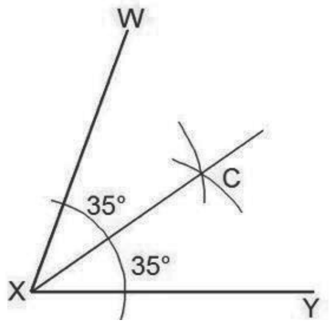
Lesson Title: Angle bisection	
Practice Activity: PHM2-L078	

1. 

2. a. 12 metres
   b. 12 metres
3. Example answer (your angle may have a different measure):

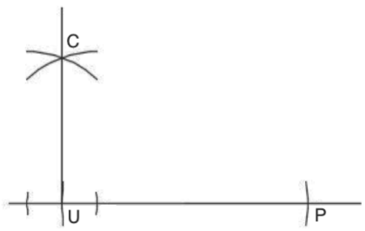


4. Example answer (your angle may have a different measure):



Lesson Title:	Angle construction
Practice Activity:	PHM4-L079

1.



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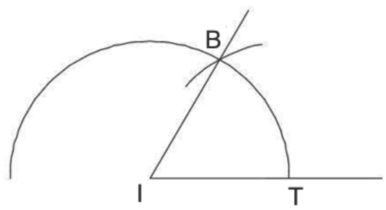
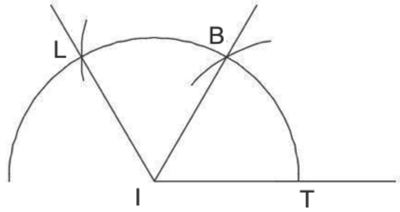


3.



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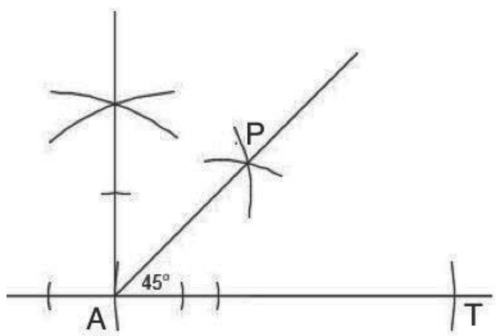
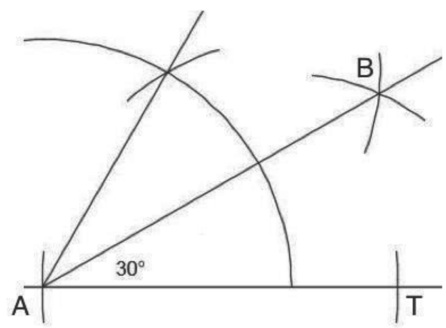


5.



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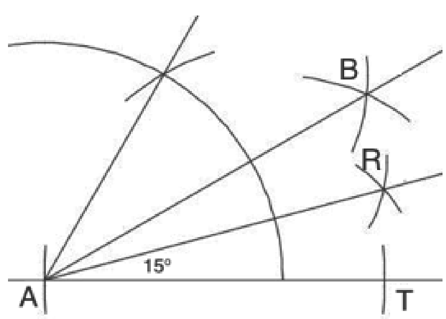

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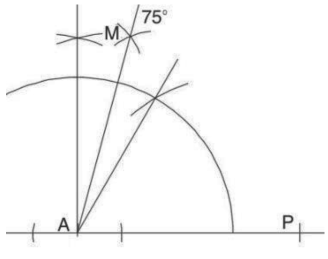
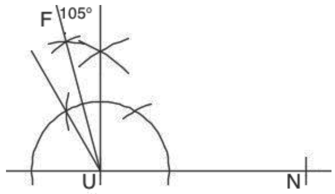


8.



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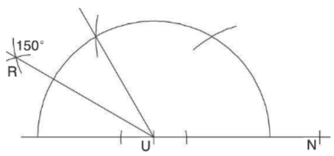
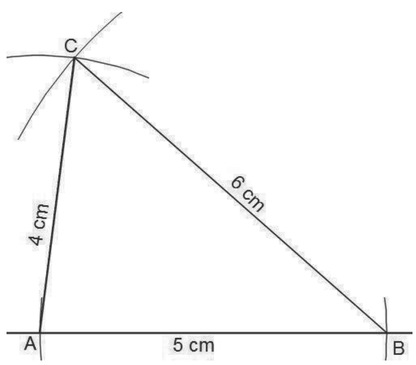


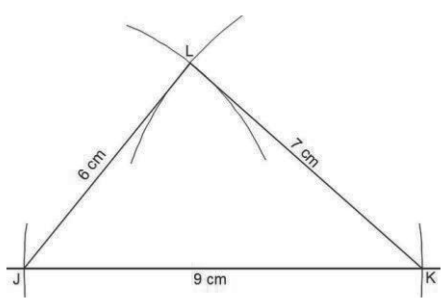
10. 135° is halfway between 90° and 180°. To construct 135°, first construct 90° and bisect the angle between 90° and 180°.

Lesson Title:	Triangle construction
Practice Activity:	PHM4-L080

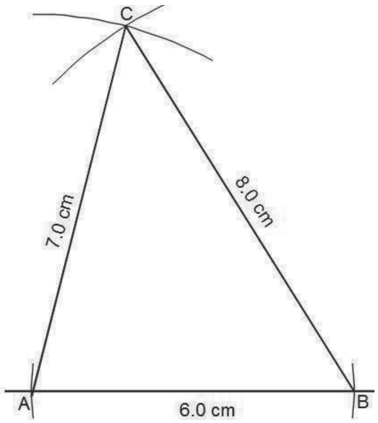
1.



2.



3.



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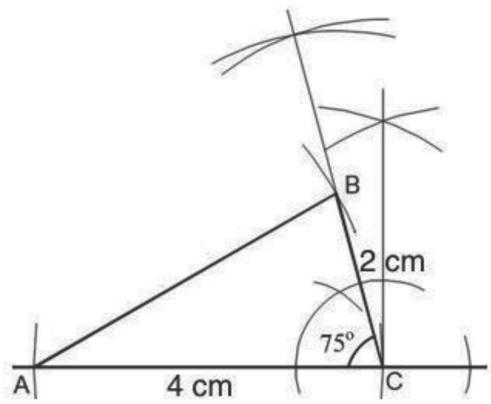

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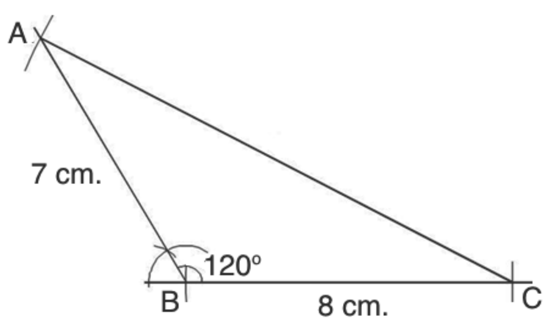

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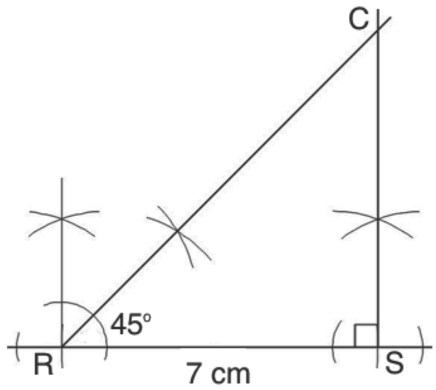

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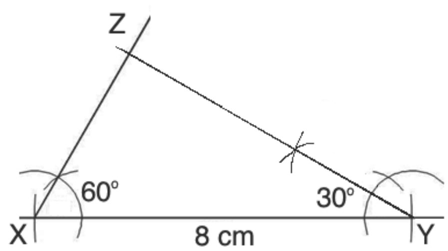

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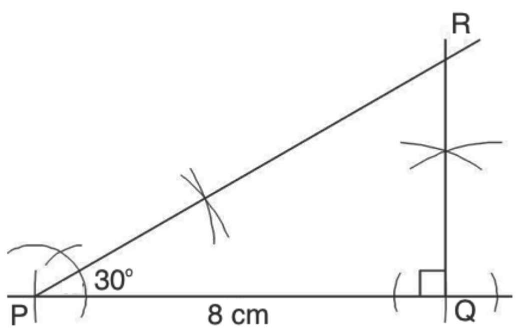
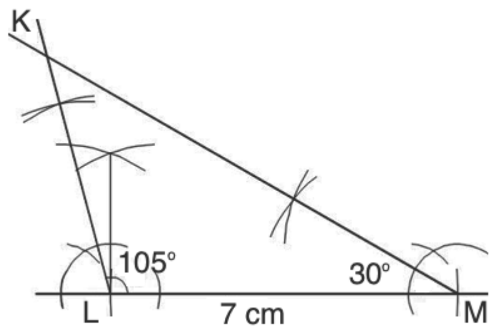
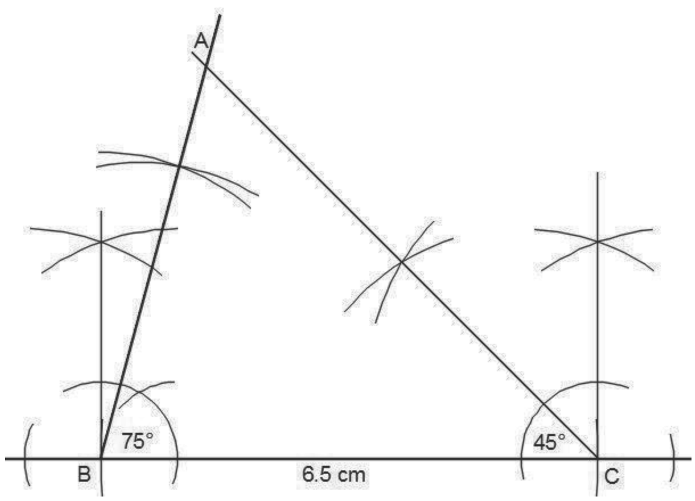


9.

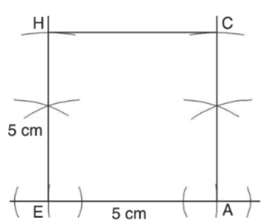


10.

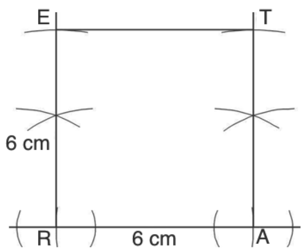


Lesson Title: Quadrilateral construction	
Practice Activity: PHM4-L081	

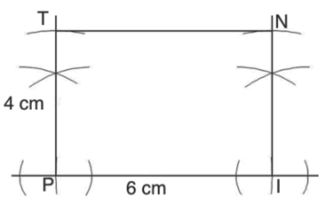
1. a.



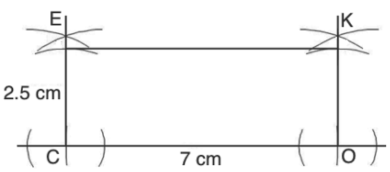
b.


E
T
6 cm
R
6 cm
A

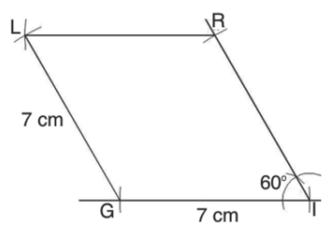
2. a.


T
N
4 cm
P
6 cm
I

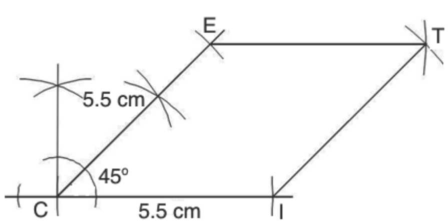
b.


E
K
2.5 cm
C
7 cm
O

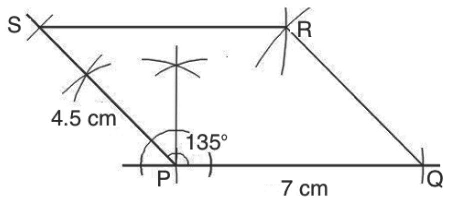
3.


L
R
7 cm
60°
G
7 cm
I

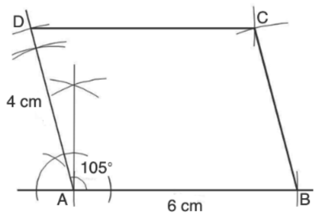
4.


E
T
5.5 cm
45°
C
5.5 cm
I

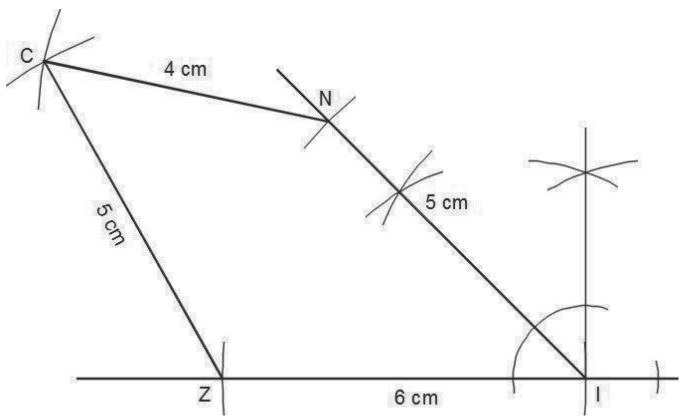
5.


S
R
4.5 cm
135°
P
7 cm
Q

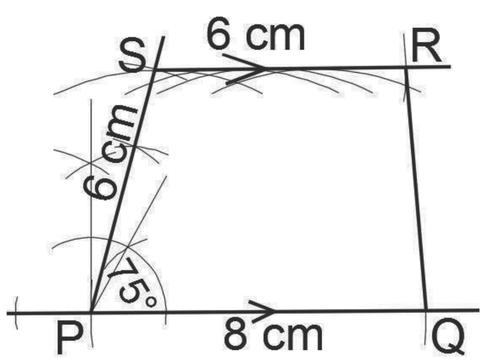
6.


D
C
4 cm
105°
A
6 cm
B

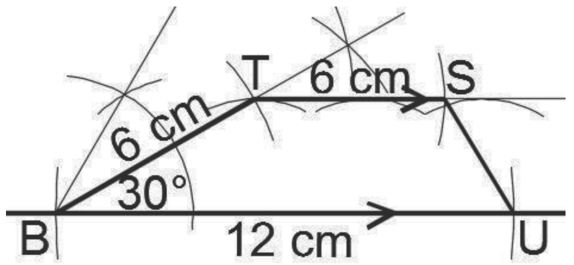
7.


C
4 cm
N
5 cm
5 cm
Z
6 cm
I

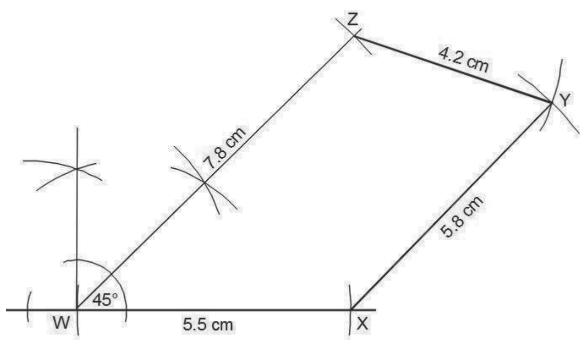
8.


S
6 cm
R
6 cm
75°
P
8 cm
Q

9.

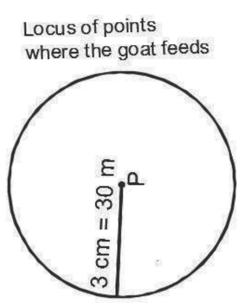


10.

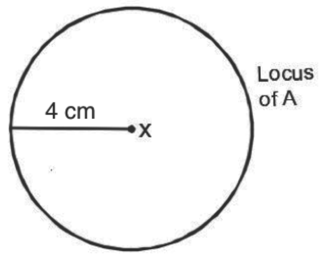


Lesson Title:	Construction of loci
Practice Activity:	PHM4-L082

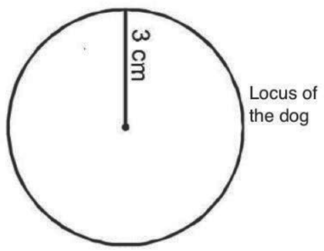
1.



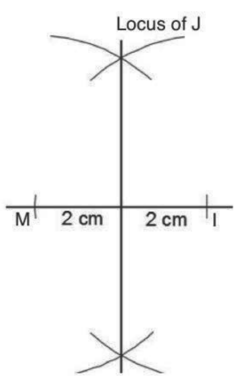
2.


Locus
of A
4 cm
x

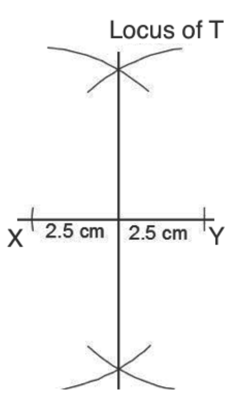
3.


3 cm
Locus of
the dog

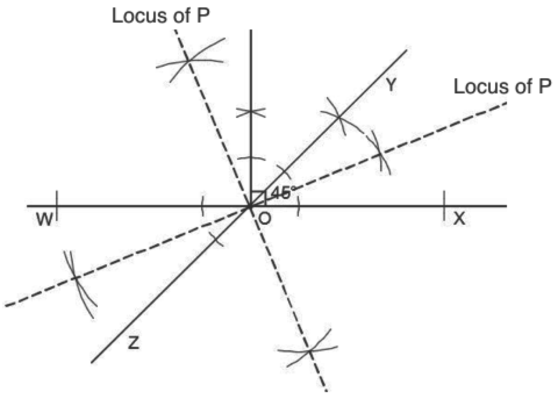
4.


Locus of J
M
2 cm
2 cm
I

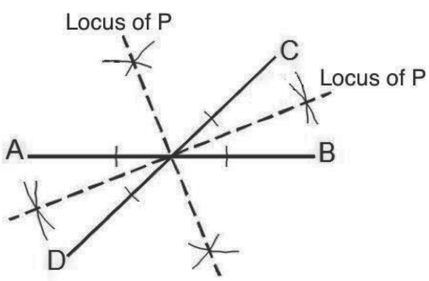
5.


Locus of T
X
2.5 cm
2.5 cm
Y

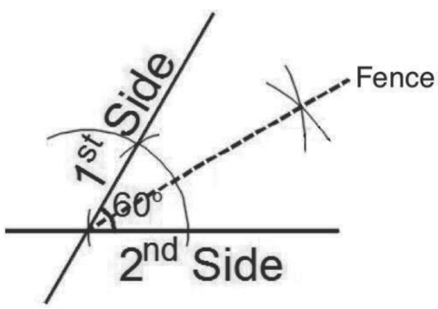
6.



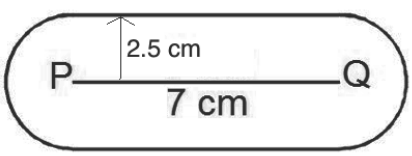
7. Example (lines may form any angle):



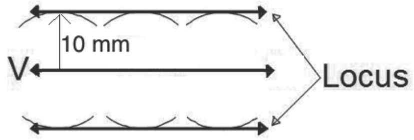
8. Solution to a and b:



9.

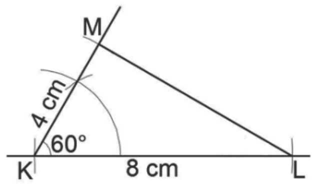


10.

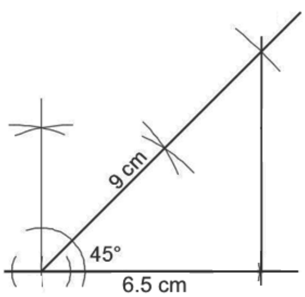


Lesson Title:	Construction word problems
Practice Activity:	PHM4-L083

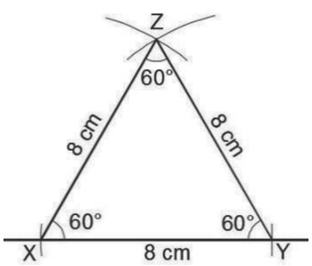
1.



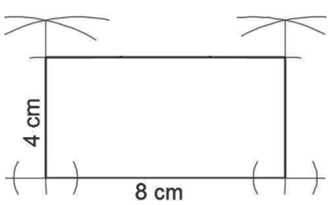
2.



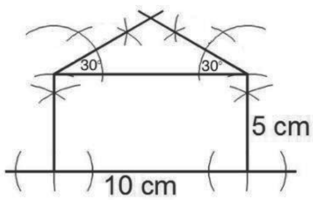
3.



4.

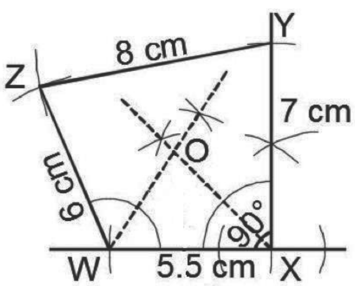


5.

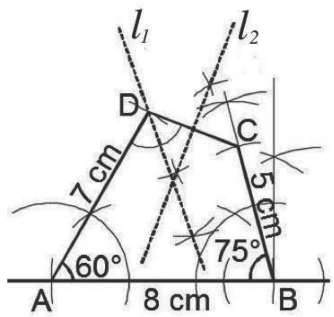


Lesson Title:	Construction of complex shapes
Practice Activity:	PHM4-L084

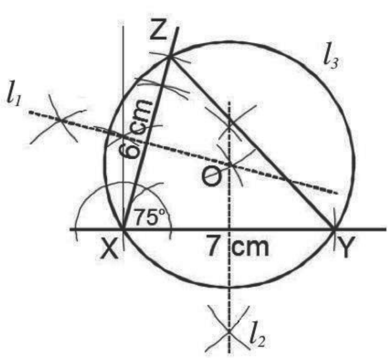
1.



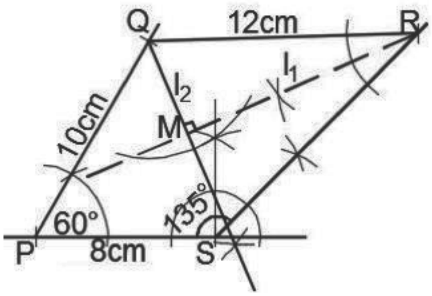
2.



3.

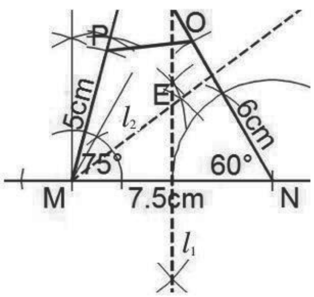


4.



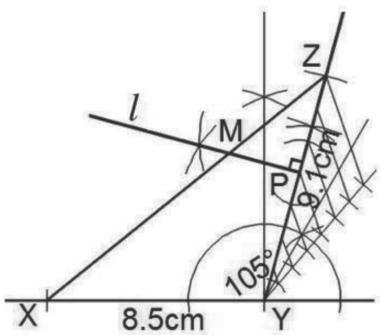
|PM|=8.1 cm; |SM|=4.75 cm

5.



|ME|=4.76 cm; |EP|=3.21 cm

6.



|YM|=5.83 cm

Lesson Title:	Addition Law of Probability
Practice Activity:	PHM4-L085

1. d
2. $P(\text{prime or even}) = \frac{13}{16}$
3. a. $P(\text{multiple of 3 or 10}) = \frac{3}{7}$; b. $P(\text{even or prime}) = \frac{17}{21}$;
   c. $P(\text{less than 12 or greater than 18}) = \frac{2}{3}$
4. a. $P(W \text{ or } R) = \frac{4}{5}$; b. $P(\bar{R}) = \frac{2}{5}$; c. $P(B \text{ or } R) = \frac{4}{5}$; d. $P(W) = \frac{1}{5}$
5. a. Completed table:

Colour	Yellow	Red	Green	Blue
Probability	0.5	0.2	0.2	0.1

   b. $P(Y \text{ or } B) = 0.6$

Lesson Title:	Multiplication law of probability
Practice Activity:	PHM4-L086

1. $\frac{1}{2}$
2. a. $\frac{30}{121}$; b. $\frac{25}{121}$; c. $\frac{36}{121}$
3. a. 0.28; b. 0.42; c. 0.12; d. 0.18
4. a. $\frac{1}{4}$; b. $\frac{1}{3}$
5. a. 52%; b. 7%
6. a. $\frac{28}{153}$ b. $\frac{2}{51}$ c. $\frac{5}{51}$ d. $\frac{8}{51}$ e. $\frac{4}{51}$ f. $\frac{16}{153}$
7. a. $\frac{14}{45}$ b. $\frac{1}{6}$ c. $\frac{7}{18}$ d. $\frac{2}{15}$

Lesson Title:	Illustration of probabilities
Practice Activity:	PHM4-L087

1. a.

Spinner B \ Spinner A	1	2	3	4
1	0	1	2	3
2	1	0	1	2
3	2	1	0	1
4	3	2	1	0

b. $\frac{4}{16} = \frac{1}{4}$ c. $\frac{2}{16} = \frac{1}{8}$ d. 0

e. $\frac{10}{16} = \frac{5}{8}$

2. a.

Packet B \ Packet A	P	B	Y	W
P	PP	BP	YP	WP
B	PB	BB	YB	WB
Y	PY	BY	YY	WY

b. $\frac{1}{12}$ c. $\frac{3}{12} = \frac{1}{4}$ d. $\frac{9}{12} = \frac{3}{4}$

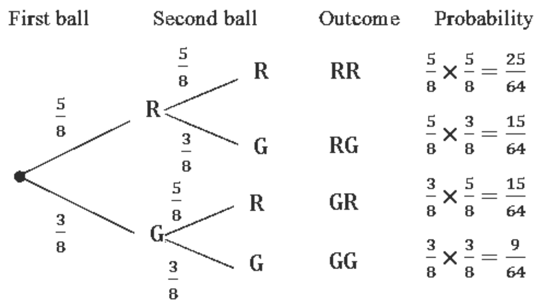
3. a.

	First day		
Second day	C	P	G
C	CC	PC	GC
P	CP	PP	GP
G	CG	PG	GG

b. $\frac{1}{9}$ c. $\frac{3}{9} = \frac{1}{3}$ d. $\frac{8}{9}$

e. $\frac{1}{9}$

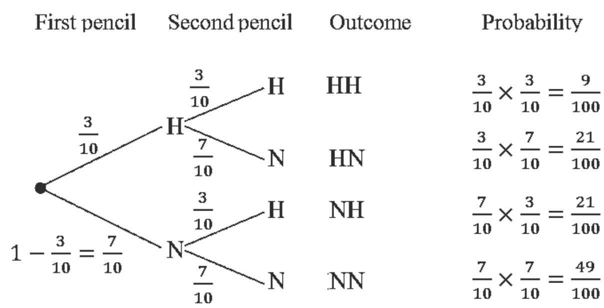
4. a.



b. $\frac{25}{64}$ c. $\frac{15}{32}$

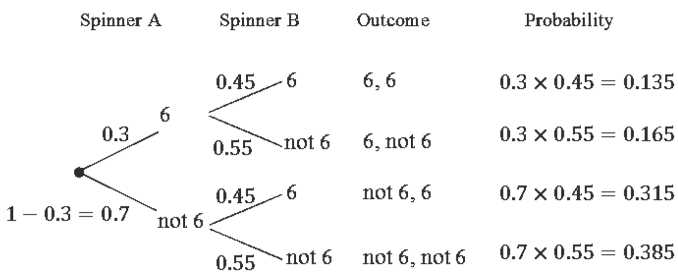
d. $\frac{55}{64}$ e. $\frac{39}{64}$

5. a. *Let* $H = HB$ *pencil and* $N =$ *not HB pencil*



b. $\frac{21}{50}$

6. a.



b. 0.385 c. 0.48

d. 0.315

Lesson Title:	Probability problem solving
Practice Activity:	PHM4-L088

1. a. 40 pupils; b. $\frac{23}{40}$; c. $\frac{11}{40}$; d. $\frac{11}{40}$
2. a. $\frac{1}{100}$; b. $\frac{5}{100} = \frac{1}{20}$
3. a. $\frac{82}{231}$; b. $\frac{149}{231}$
4. a. $\frac{4}{36} = \frac{1}{9}$; b. $\frac{6}{36} = \frac{1}{6}$; c. $\frac{14}{36} = \frac{7}{18}$
5. $\frac{40}{81}$
6. a. $\frac{22}{714}$; b. $\frac{44}{1785}$; c. $\frac{209}{1785}$
7. 24 males
8. $\frac{9}{26}$
9. a. $\frac{1}{3}$; b. $\frac{1}{3}$; c. 0

Answer Key – Last 8 lessons (Mock Exam)

Lesson Title: Mock Examination: Paper 1 – Multiple Choice	
Practice Activity: PHM4-L089	

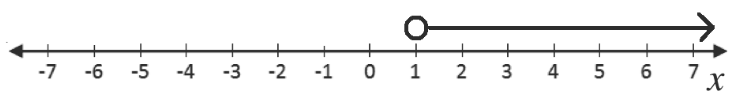
1. Answer: C. 2,916
   Solution:
$$\begin{aligned} U_7 &= ar^{n-1} && \text{Formula for } n\text{th term of a GP} \\ &= 4(3^{7-1}) && \text{Substitute } a, n, \text{ and } r \\ &= 4(3^6) && \text{Simplify} \\ &= 2{,}916 \end{aligned}$$

2. Answer: B. $x^2 - x - 2 = 0$
   Solution:
$$\begin{aligned} x^2 + 2x - 3 &= 3x - 1 && \text{Set equations for } y \text{ equal} \\ x^2 + 2x - 3x - 3 + 1 &= 0 && \text{Write in standard form} \\ x^2 - x - 2 &= 0 && \text{Simplify} \end{aligned}$$

3. Answer: B. $173°$
   Solution:
   Girls $= \frac{300}{625} \times 360° = 172.8° = 173°$ to the nearest degree.

4. Answer: C. 12
   Solution:
$$\begin{aligned} \bar{x} &= \frac{\sum fx}{\sum f} && \text{Formula for mean of grouped data} \\ &= \frac{3(1)+8(2)+13(5)+18(2)}{1+2+5+2} && \text{Substitute values} \\ &= \frac{120}{10} && \text{Simplify} \\ &= 12 \text{ marks} \end{aligned}$$

5. Answer: B.



   Solution:
$$\begin{aligned} \frac{2x}{3} - \frac{5}{6} &> -\frac{x}{6} \\ 4x - 5 &> -x && \text{Multiply by the LCM, 6} \\ -5 &> -x - 4x && \text{Transpose } 4x \\ -5 &> -5x \\ \frac{-5}{-5} &< \frac{-5x}{-5} && \text{Divide throughout by -5 (inequality changes)} \\ 1 &< x \end{aligned}$$

6. Answer: A. $a = \frac{b^2+5}{b^2-2}$

   Solution:

$$
\begin{aligned}
b &= \sqrt{\frac{2a+5}{a-1}} \\
b^2 &= \frac{2a+5}{a-1} && \text{Square both sides} \\
b^2(a-1) &= 2a+5 && \text{Multiply both sides by } (a-1) \\
ab^2 - b^2 &= 2a+5 \\
ab^2 - 2a &= b^2+5 && \text{Collect terms with } a \text{ on one side} \\
a(b^2-2) &= b^2+5 && \text{Factor out } a \\
a &= \frac{b^2+5}{b^2-2} && \text{Divide both sides by } (b^2-2)
\end{aligned}
$$

7. Answer: D. $R' \cap (P \cap Q)$

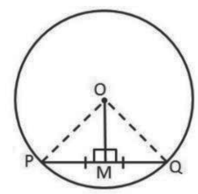
   Solution:

   $R' \cap (P \cap Q)$ gives the elements in the intersection of P and Q, which are not also in set R.

8. Answer: B. 20 cm

   Solution:

   Recall that the distance of the chord from the centre of the circle is the perpendicular bisector of the chord. Using the diagram on the right, we are given |OM|=16 cm and |PQ|=24 cm. Therefore, we have $|\mathrm{PM}| = \frac{1}{2}|\mathrm{PQ}| = \frac{1}{2}(24) = 12$. Apply Pythagoras' theorem to find the radius, OP.



$$
\begin{aligned}
OP^2 &= PM^2 + OM^2 \\
r^2 &= 12^2 + 16^2 \\
r^2 &= 144 + 256 \\
r^2 &= 400 \\
r &= \sqrt{400} = 20 \text{ cm}
\end{aligned}
$$

9. Answer: C. 22

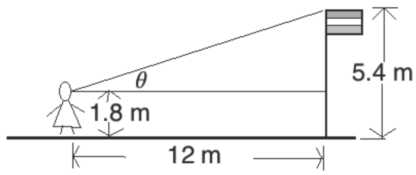
   Solution:

$$
\begin{aligned}
\frac{1}{2}x + 3x &= 77 && \text{Equation with } x \text{ as the unknown number} \\
x + 6x &= 154 && \text{Multiply throughout by 2} \\
7x &= 154 && \text{Simplify} \\
x &= \frac{154}{7} = 22
\end{aligned}
$$

10. Answer: A. 17°

Solution:

Using the diagram on the right,



$$\tan\theta = \frac{5.4-1.8}{12}$$
$$\tan\theta = \frac{3.6}{12}$$
$$\tan\theta = 0.3$$
$$\theta = \tan^{-1} 0.3$$
$$\theta = 17° \text{ to the nearest degree}$$

11. Answer: D. 8

Solution:

$$135° = \frac{(n-2)\times 180°}{n}$$ Formula for interior angle of a regular polygon
$$135°n = (n-2)\times 180°$$ Solve for n
$$135°n = 180°n - 360°$$
$$360° = 180°n - 135°n$$
$$360° = 45°n$$
$$\frac{360°}{45°} = n$$
$$n = 8$$

12. Answer: A. 3 days

Solution:

If 20 people eat a bag of rice in 12 days, then 1 person eats the same bag of rice in $20 \times 12 = 240$ days.

Divide 240 days by 80 people: $240 \div 80 = 3$ days

13. Answer: A. 0

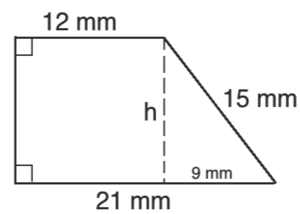
Solution:

$$\frac{a}{2a+4b} + \frac{b}{a+2b} - \frac{1}{2} = \frac{a}{2(a+2b)} + \frac{b}{a+2b} - \frac{1}{2}$$ Factor the denominators
$$= \frac{a}{2(a+2b)} + \frac{2b}{2(a+2b)} - \frac{a+2b}{2(a+2b)}$$ Change denominators to the LCM
$$= \frac{a+2b-(a+2b)}{2(a+2b)}$$ Add/Subtract
$$= \frac{a+2b-a-2b}{2(a+2b)}$$ Remove brackets
$$= \frac{0}{2(a+2b)} = 0$$ Simplify

14. Answer: A. 198 mm^2

Solution:

Use Pythagoras' theorem to find the height of the trapezium.



$$
\begin{aligned}
h^2 + 9^2 &= 15^2 \\
h^2 + 81 &= 225 \\
h^2 &= 225 - 81 = 144 \\
h &= \sqrt{144} = 12 \text{ mm}
\end{aligned}
$$

Apply the formula for area of a trapezium:

$$
\begin{aligned}
A &= \frac{1}{2}(a+b)h \\
&= \frac{1}{2}(12+21)12 \\
&= 198 \text{ mm}^2
\end{aligned}
$$

15. Answer: B. $1\frac{1}{2}$

Solution:

$$
\begin{aligned}
36^{\frac{1}{2}} \times 8^{-\frac{2}{3}} &= \sqrt{36} \times \frac{1}{8^{\frac{2}{3}}}. && \text{Simplify} \\
&= 6 \times \frac{1}{\sqrt[3]{8^2}} \\
&= 6 \times \frac{1}{\sqrt[3]{64}} \\
&= 6 \times \frac{1}{4} \\
&= \frac{6}{4} = 1\frac{1}{2} && \text{Multiply}
\end{aligned}
$$

16. Answer: C. 198 m^2

Using the ratio 2 : 1 : 3, let $l = 2x$, $w = x$ and $h = 3x$ for some value of x.

Solve for x using volume:

$$
\begin{aligned}
V &= l \times w \times h \\
162 &= (2x)(x)(3x) \\
162 &= 6x^3 \\
\frac{162}{6} &= x^3 \\
27 &= x^3 \\
\sqrt{27} &= x \\
3 &= x
\end{aligned}
$$

Therefore, the dimensions are: $l = 2(3) = 6$ m, $w = 3$ m and $h = 3(3) = 9$ m.

Apply the formula for surface area of a cuboid:

$$
\begin{aligned}
SA &= 2(lh + hw + lw) \\
&= 2(6 \times 9 + 9 \times 3 + 6 \times 3) \\
&= 2(54 + 27 + 18) \\
&= 198 \text{ m}^2
\end{aligned}
$$

17. Answer: C. ₦15,000.00

Fraction of money saved: $1 - \frac{1}{2} - \frac{1}{5} = \frac{10}{10} - \frac{5}{10} - \frac{2}{10} = \frac{3}{10}$

Set up an equation with the total amount, call it A:

$$\frac{3}{10}A = ₦9{,}000$$
$$3A = ₦90{,}000$$
$$A = ₦\frac{90{,}000}{3} = ₦30{,}000.00$$

Amount spent on food: $\frac{1}{2}(30{,}000) =$ ₦ 15,000.00

18. Answer: B. 72°

Solution:

Using circle theorems, we know that $\angle ABC + \angle ADC = 180°$ because they are opposite angles in a cyclical quadrilateral. We also know that $2\angle ABC = \angle AOC$, because the same chord subtends these 2 angles at the circumference and centre. Using these theorems, we can write 2 equations for $\angle ABC$ in x:

$$\angle ABC + \angle ADC = 180° \rightarrow \angle ABC = 180° - \angle ADC \rightarrow \angle ABC = 180° - 3x$$
$$2\angle ABC = \angle AOC \rightarrow \angle ABC = \frac{1}{2}\angle AOC \rightarrow \angle ABC = \frac{1}{2}(4x) = 2x$$

Set the 2 equations for $\angle ABC$ equal, and solve for x:

$$180° - 3x = 2x$$
$$180° = 2x + 3x$$
$$180° = 5x$$
$$\frac{180°}{5} = x$$
$$36° = x$$

Substitute $x = 36°$ into either formula for $\angle ABC$.

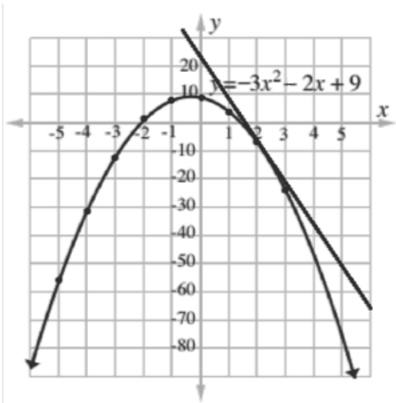
$$\angle ABC = 2x = 2(36°) = 72°$$

Lesson Title: Mock Examination: Paper 1 – Multiple Choice
Practice Activity: PHM4-L090

1. Answer: D. $\{t, u, v, w, y\}$
   $(A \cap B)'$ is the complement of the intersection of A and B. In other words, it is the elements in the universal set U which are not in the intersection of A and B. First, find the intersection of A and B: $A \cap B = \{x, z\}$. Next, list the elements of U that are not in $A \cap B$. This gives: $(A \cap B)' = \{t, u, v, w, y\}$.
2. Answer: A. $x = \frac{3y}{z}$
   x varies directly as y and inversely as z is written in symbols as $x \propto \frac{y}{z}$. This can also be written as an equation $x = \frac{ky}{z}$, where k is a constant. Use the information in the problem to solve for k:

$$\begin{aligned} x &= \frac{ky}{z} && \text{Equation} \\ 2 &= \frac{k8}{12} && \text{Substitute } x = 2,\ y = 8 \text{ and } z = 12 \\ 2(12) &= 8k \\ 24 &= 8k \\ \frac{24}{8} &= k \\ 3 &= k \end{aligned}$$

   Therefore, the relationship between x, y and z is $x = \frac{3y}{z}$.
3. Answer: B. -2.1
   The roots are points at which the curve intersects the x-axis. The smaller root is the negative one, which intersects the curve near $x = -2$. Select the answer nearest to -2, which is -2.1.
4. Answer: C. -14
   Sketch a tangent to the curve at $x = 2$, and use it to estimate the slope. Note that the slope is negative, and the y-axis has a scale of 10 units. At this point, you could eliminate answers that are not feasible (such as positive 14).
   You can also apply the formula for gradient. Choose any 2 points on the tangent line, and use whole or rounded numbers to save time. Points $(5, -50)$ and $(0, 20)$ are on or near the tangent line.



$$\begin{aligned} \text{Gradient: } m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{20 - (-50)}{0 - 5} \\ &= -\frac{70}{5} \end{aligned}$$

$$= \quad -14$$

5. Answer: B. 6.5 m.
   From the circle theorems, recall that an angle subtended in a circle by the diameter is a right angle. Thus, ABC is a right-angled triangle. Apply Pythagoras' theorem to find the diameter:
$$\begin{aligned} AC^2 &= AB^2 + BC^2 \\ AC^2 &= 5^2 + 12^2 \\ AC^2 &= 25 + 144 \\ AC^2 &= 169 \\ AC &= \sqrt{169} = 13 \text{ m} \end{aligned}$$
   Use the diameter to find the radius: $r = \frac{d}{2} = \frac{13}{2} = 6.5$ m

6. Answer: A. 19.5 m^2
   Note that there is a 0.75-metre margin on each side of the carpet. Thus, subtract twice that from each dimension before finding the area.
   Length $= 8 - 2(0.75) = 8 - 1.5 = 6.5$ m
   Width $= 4.5 - 2(0.75) = 4.5 - 1.5 = 3$
   Area $= l \times w = 6.5 \times 3 = 19.5$ m^2
7. Answer: B. 20 cm
   Note that XWV and XYZ are similar triangles. Therefore, their sides are proportional, and a ratio can be created with $|VW|$ and known side lengths:
   $\frac{24 \text{ cm}}{48 \text{ cm}} = \frac{|VW|}{40 \text{ cm}}$
   Simplifying this, we have $\frac{1 \text{ cm}}{2 \text{ cm}} = \frac{|VW|}{40 \text{ cm}}$. Solve for $|VW|$:
$$\begin{aligned} \frac{|VW|}{40 \text{ cm}} &= \frac{1 \text{ cm}}{2 \text{ cm}} \\ 2|VW| &= 40 \times 1 \\ 2|VW| &= 40 \\ |VW| &= \frac{40}{2} \\ |VW| &= 20 \text{ cm} \end{aligned}$$
8. Answer: D. $2x^2 + 5x - 3 = 0$
   To find a quadratic equation given its roots, find b and c of the quadratic equation in standard form. This can be done by finding the sum and product of the roots and substituting them in the following equation:
$$x^2 + bx + c = x^2 - (sum\ of\ roots)x + (product\ of\ roots) = 0$$
   Sum of roots: $-3 + \frac{1}{2} = -2\frac{1}{2} = -\frac{5}{2}$
   Product of roots: $-3 \times \frac{1}{2} = -\frac{3}{2}$
   Quadratic equation: $x^2 - \left(-\frac{5}{2}\right)x + \left(-\frac{3}{2}\right) = x^2 + \frac{5}{2}x - \frac{3}{2} = 0$
   Multiply throughout by 2 to eliminate fractions: $2x^2 + 5x - 3 = 0$

9. Answer: D. $(x+7y)(2a-3b)$

Rearrange the terms and factorize as follows:

$2ax-21by-3bx+14ay$	$=$	$2ax-3bx-21by+14ay$	Rearrange
	$=$	$x(2a-3b)+7y(-3b+2a)$	Factorize
	$=$	$x(2a-3b)+7y(2a-3b)$	
	$=$	$(x+7y)(2a-3b)$	

10. Answer: A. $-1\frac{2}{3}$

Substitute the given values into the formula and evaluate:

$\frac{a}{b}+\frac{3x}{a}-1\frac{1}{2}$	$=$	$\frac{4}{3}+\frac{3(-2)}{4}-1\frac{1}{2}$	
	$=$	$\frac{4}{3}-\frac{6}{4}-\frac{3}{2}$	Simplify
	$=$	$\frac{16}{12}-\frac{18}{12}-\frac{18}{12}$	Change denominators to the LCM, 12
	$=$	$\frac{16-18-18}{12}$	Subtract
	$=$	$\frac{-20}{12}$	Simplify
	$=$	$\frac{-5}{3}=-1\frac{2}{3}$	

11. Answer: B. $4(4+\pi)$

Note that the perimeter is composed of 2 radii and an arc. We already know the radius of the circle. Find the length of the arc:

$\frac{90°}{360°}C$	$=$	$\frac{1}{4}2\pi r=\frac{1}{2}\pi r$	Simplify formula
	$=$	$\frac{1}{2}\pi(8)$	Substitute $r=8$ cm
	$=$	4π	

Add to find the perimeter:

P	$=$	$8+8+4\pi$	Simplify formula
	$=$	$16+4\pi$	Substitute $r=8$ cm
	$=$	$4(4+\pi)$	Factorise

12. Answer: C. $\frac{1}{2}$

Convert each term to an index with base 2, then apply the laws of logarithms to simplify.

$\frac{8^x \times 2^{x+1}}{4^{3x}}$	$=$	1	
$\frac{2^{3x} \times 2^{x+1}}{2^{2\times 3x}}$	$=$	1	Convert to base 2
$\frac{2^{3x} \times 2^{x+1}}{2^{6x}}$	$=$	1	Simplify
$\frac{2^{3x+x+1}}{2^{6x}}$	$=$	1	Apply law of multiplication of logarithms
$\frac{2^{4x+1}}{2^{6x}}$	$=$	1	Simplify
$2^{4x+1-6x}$	$=$	1	
2^{-2x+1}	$=$	1	Note that $a^0 = 1$
2^{-2x+1}	$=$	2^0	
$-2x + 1$	$=$	0	Set the powers equal
$-2x$	$=$	-1	Solve for x
x	$=$	$\frac{-1}{-2} = \frac{1}{2}$	

13. Answer: B. 5

Convert both sides to base 10, then set them equal and solve for x.

Convert the left hand side from base x to base 10:

$$34_x = (3 \times x^1) + (4 \times x^0)$$
$$= 3x + 4$$

Convert the right hand side from base 2 to base 10:

$$10011_2 = (1 \times 2^4) + (0 \times 2^3) + (0 \times 2^2) + (1 \times 2^1) + (1 \times 2^0)$$
$$= 16 + 0 + 0 + 2 + 1$$
$$= 19$$

Set the two sides equal and solve for x:

$3x + 4$	$=$	19	
$3x$	$=$	$19 - 4$	Transpose 4
$3x$	$=$	15	
x	$=$	$\frac{15}{3}$	Divide throughout by 3
x	$=$	5	

14. Answer: A. 2.5 cm

Use the area of the sector to find the angle subtended by the arc of the circle. The length of the arc will be the circumference of the base of the cone. Use the formula for circumference to solve for the radius of the base.

A	$=$	$\frac{\theta}{360°}\pi r^2$	Area of a segment
25π	$=$	$\frac{\theta}{360°}\pi 10^2$	
25	$=$	$\frac{100}{360°}\theta$	Cancel π and simplify
25	$=$	$\frac{5}{18}\theta$	
25×18	$=$	5θ	
450	$=$	5θ	
$\frac{450}{5}$	$=$	θ	
$90°$	$=$	θ	

Length of arc:

L	$=$	$\frac{\theta}{360°}2\pi r$	Length of an arc
L	$=$	$\frac{90°}{360°}2\pi(10)$	Substitute $\theta = 90°$ and $r = 10$
L	$=$	$\frac{1}{4}20\pi$	Simplify
L	$=$	5π	

Since the length of the arc is the circumference of the base, apply the formula for circumference:

C	$=$	$2\pi r$	Circumference of a circle
5π	$=$	$2\pi r$	Substitute $C = 5\pi$
$\frac{5\pi}{2\pi}$	$=$	r	Solve for r
$\frac{5}{2}$	$=$	r	
2.5	$=$	r	

15. Answer: C. 9

Set up simultaneous equations and solve using substitution. Let x be the number of Le5,000 notes and y be the number of Le1,000 notes. Then we have:

$$\begin{aligned} x + y &= 16 && \text{Equation (1)} \\ 5{,}000x + 1{,}000y &= 52{,}000 && \text{Equation (2)} \end{aligned}$$

$$\begin{aligned} y &= 16 - x && \text{Change the subject of (1)} \\ 5{,}000x + 1{,}000(16 - x) &= 52{,}000 && \text{Substitute (1) in (2)} \\ 5{,}000x + 16{,}000 - 1{,}000x &= 52{,}000 && \text{Solve for } x \\ 4{,}000x + 16{,}000 &= 52{,}000 \\ 4{,}000x &= 52{,}000 - 16{,}000 \\ 4{,}000x &= 36{,}000 \\ x &= 9 \end{aligned}$$

He has 9 Le 5,000 notes.

16. Answer: A. 29

Note that in order to solve for y, we must first solve for x. Use the angles below the line AB to find the measure of x.

$$\begin{aligned} 180° &= 30° + 90° + (2x° - 10°) && \text{Set equal to } 180° \\ 180° &= 30° + 90° + 2x° - 10° && \text{Solve for } x \\ 180° &= 110° + 2x° \\ 180° - 110° &= 2x° \\ 70° &= 2x° \\ \tfrac{70°}{2} &= x° \\ 35° &= x \end{aligned}$$

Use the angles above the line AB to solve for y:

$$\begin{aligned} 180° &= 5y° + x° && \text{Set equal to } 180° \\ 180° &= 5y° + 35° && \text{Solve for } y \\ 180° - 35° &= 5y° \\ 145° &= 5y° \\ \tfrac{145°}{5} &= y° \\ 29° &= y \end{aligned}$$

17. Answer: B. 18.2%

Calculate percentage error using the formula:

$$\begin{aligned}\text{Percentage error} &= \frac{|\text{exact value} - \text{approximate value}|}{\text{exact value}} \times 100\% \\ &= \frac{|18-22\,|}{22} \times 100\% \\ &= \frac{4}{22} \times 100\% \\ &= 18.2\%\end{aligned}$$

18. Answer: D. 16.7 m/s

Use the information in the problem to find her spend in km/hour:

Speed $= \frac{240\text{ km}}{4\text{ hr}} = 60$ km/hr

Use the conversion factors 1 hour = 3,600 seconds, and 1 km = 1,000 metres to convert her speed to m/s.

Speed $= \frac{60\text{ km}}{1\text{ hr}} \times \frac{1000\text{ m}}{1\text{ km}} \times \frac{1\text{ hr}}{3600\text{ s}} = \frac{600}{36} = 16.7$ m/s

Lesson Title: Mock Examination: Paper 2A – Compulsory Questions	
Practice Activity: PHM4-L091	

1. a. Apply the correct order of operations (BODMAS):

$$\frac{1\frac{1}{2}+2\frac{1}{3}}{2\frac{1}{2}-3\frac{3}{4}\times\frac{2}{5}} = \frac{\frac{3}{2}+\frac{7}{3}}{\frac{5}{2}-\frac{15}{4}\times\frac{2}{5}} \quad \text{Convert to improper fractions}$$

$$= \frac{\frac{3}{2}+\frac{7}{3}}{\frac{5}{2}-\frac{30}{20}} \quad \text{Multiply}$$

$$= \frac{\frac{9}{6}+\frac{14}{6}}{\frac{5}{2}-\frac{3}{2}}$$

$$= \frac{\frac{23}{6}}{\frac{2}{2}} \quad \text{Add/Subtract}$$

$$= \frac{3\frac{5}{6}}{1} \quad \text{Simplify}$$

$$= 3\frac{5}{6}$$

b. Multiply, then simplify:

$$\begin{aligned}(\sqrt{2}-3\sqrt{5})(\sqrt{2}+\sqrt{5}) &= \sqrt{2}(\sqrt{2}+\sqrt{5})-3\sqrt{5}(\sqrt{2}+\sqrt{5})\\ &= \sqrt{2}\sqrt{2}+\sqrt{2}\sqrt{5}-3\sqrt{2}\sqrt{5}-3\sqrt{5}\sqrt{5}\\ &= 2+\sqrt{10}-3\sqrt{10}-3(5)\\ &= 2-15-2\sqrt{10}\\ &= -13-2\sqrt{10}\end{aligned}$$

Answer: $a=-13$, $b=-2$

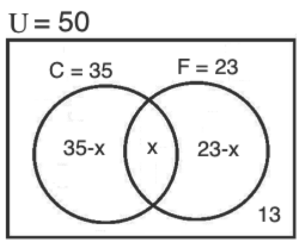
2. a. i. mode: 3 children

ii. mean $=\frac{\sum fx}{\sum f}=\frac{0(1)+1(4)+2(3)+3(5)+4(4)+5(3)}{1+4+3+5+4+3}=\frac{4+6+15+16+15}{20}=\frac{56}{20}=2.8$ children

iii. The position of the 3rd quartile is the $\frac{3}{4}(20)=15$th family. The 15th family falls into the group with 4 children. Thus, the 3rd quartile is 4 children.

b. The segment representing 4 children is given by: $\frac{4}{20}(360°)=72°$.

3. a. i. Venn diagram:



ii. Set the sum of the segments equal to 50, the total number of pupils, and solve for x.

$$\begin{aligned}
50 &= (35 - x) + x + (23 - x) + 13 \\
50 &= 35 + 23 + 13 - x \\
50 &= 71 - x \\
x &= 71 - 50 \\
x &= 21
\end{aligned}$$

iii. To find the cardinality of the union, add the cardinality of each set and subtract the elements in their intersection.

$$\begin{aligned}
n(C \cup F) &= n(C) + n(F) - n(C \cap F) \\
&= 35 + 23 - 21 \\
&= 37
\end{aligned}$$

Alternatively, identify the cardinality of each section of the union of C and F, and add them:

$$\begin{aligned}
n(C \cup F) &= 35 - x + x + 23 - x \\
&= 58 - x \\
&= 58 - 21 \\
&= 37
\end{aligned}$$

b. Substitute a and b in the expression, and evaluate.

$$\begin{aligned}
\frac{a+b}{a-b} &= \frac{\frac{2x}{x+1}+\frac{x-1}{x+1}}{\frac{2x}{x+1}-\frac{x-1}{x+1}} \\
&= \frac{\frac{2x+x-1}{x+1}}{\frac{2x-x+1}{x+1}} \\
&= \frac{\frac{3x-1}{x+1}}{\frac{x+1}{x+1}} \\
&= \frac{\frac{3x-1}{x+1}}{1} \\
&= \frac{3x-1}{x+1}
\end{aligned}$$

Lesson Title: Mock Examination: Paper 2A – Compulsory Questions	
Practice Activity: PHM4-L092	

1. a. Change the subject:

$$
\begin{aligned}
q &= \sqrt{t^2p - \frac{rp}{t}} & \\
q^2 &= t^2p - \frac{rp}{t} & \text{Square both sides} \\
q^2 &= \frac{t^3p - rp}{t} & \text{Subtract the right-hand side} \\
q^2t &= t^3p - rp & \text{Multiply throughout by } t \\
q^2t &= p(t^3 - r) & \text{Factor } p \text{ from the right-hand side} \\
\frac{q^2t}{t^3 - r} &= p & \text{Divide throughout by } (t^3 - r)
\end{aligned}
$$

b. Write the first equation with indices of base 2 on both sides of the equation:

$$2^{x+1} = 8^{3y} \rightarrow 2^{x+1} = 2^{3(3y)} \rightarrow 2^{x+1} = 2^{9y}$$

Set the exponents equal: $x + 1 = 9y$

Now we have simultaneous linear equations: $x + 1 = 9y$ and $2x + y = 36$. Solve using substitution:

$$
\begin{aligned}
x + 1 &= 9y & (1) & \\
x &= 9y - 1 & & \text{Change the subject of equation (1)} \\
\\
2(9y - 1) + y &= 36 & (1) & \text{Substitute equation (1) into equation (2)} \\
18y - 2 + y &= 36 & & \text{Simplify the left-hand side} \\
19y - 2 &= 36 & & \\
19y &= 38 & & \text{Add 2 throughout} \\
\frac{19y}{19} &= \frac{38}{19} & & \text{Divide throughout by } 19 \\
y &= 2 & &
\end{aligned}
$$

Substitute y into either equation to find x: $x = 9y - 1 = 9(2) - 1 = 18 - 1 = 17$

$x = 17, y = 2$

Therefore, $x - y = 17 - 2 = 15$.

2. a. Note that the total number of pupils is 45, and the probability of selecting a male is $1 - \frac{1}{3} = \frac{2}{3}$. Multiply this by the number of pupils: $\frac{2}{3}(45) = 30$

   Answer: There are 30 male pupils.

   b. The probability of selecting a female pupil is $\frac{1}{3}$, so the number of female pupils currently is $\frac{1}{3} \times 45 = 15$. A certain number of females should be added to the total to create a probability of $\frac{1}{2}$. Let's call that number f. Then we have: $\frac{1}{2} =$

$\frac{15+f}{45+f}$. This is based on the new probability, and adding f to both the total number, and the number of females.

$$\begin{aligned} \frac{1}{2} &= \frac{15+f}{45+f} \\ 45+f &= 2(15+f) && \text{Cross-multiply} \\ 45+f &= 30+2f \\ 45-30 &= 2f-f \\ 15 &= f \end{aligned}$$

Answer: 15 more female pupils should be enrolled.

3. a. Note that $\angle OBA = 90°$ because a tangent line is perpendicular to the radius. Therefore, we can subtract $\angle OBC$ from $90°$ to find $\angle ABC$. Find $\angle OBC$ using the triangle formed by the chord, which forms a perpendicular angle with AO.

$$\angle OBC = 180° - 90° - 65° = 25°$$

Therefore, $\angle ABC = 90° - \angle OBC = 90° - 25° = 65°$

Note that $\angle OAB$ can be found using $\triangle\ OAB$. Subtract the known angles from $180°$: $\angle OAB = 180° - 90° - 65° = 25°$

Answers: $\angle ABC = 65°$, $\angle OAB = 25°$

b. Note that the interior angles of a hexagon sum to $720°$. Sum the interior angles and solve x:

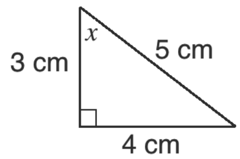
$$\begin{aligned} 720° &= 5x+5x+6x+6x+7x+7x \\ 720° &= 36x \\ \frac{720°}{36} &= x \\ 20° &= x \end{aligned}$$

Lesson Title: Mock Examination: Paper 2A – Compulsory Questions	
Practice Activity: PHM4-L093	

1. a. Expand each part of the expression, then simplify:

$$\begin{aligned}(3x+y)^2-(y-2x)^2 &= 3x(3x+y)+y(3x+y)-y(y-2x)+2x(y-2x)\\ &= 9x^2+3xy+3xy+y^2-y^2+2xy+2xy-4x^2\\ &= 5x^2+10xy\\ &= 5x(x+2y)\end{aligned}$$

   b. Draw a right-angled triangle using (see below). Use Pythagoras' theorem to find the third side, which is 3.

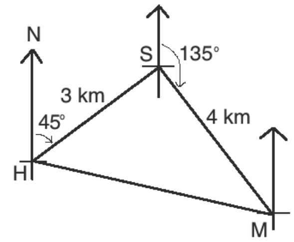


   Use the triangle to find the values of $\cos x$ and $\tan x$, which are needed for the formula in the problem. $\cos x=\frac{3}{5}$; $\tan x=\frac{4}{3}$.

   Substitute each trigonometric ratio into the formula and simplify:

$$\begin{aligned}\frac{2\cos x-3\sin x}{\tan x} &= \frac{2\left(\frac{3}{5}\right)-3\left(\frac{4}{5}\right)}{\frac{4}{3}}\\ &= \frac{\frac{6}{5}-\frac{12}{5}}{\frac{4}{3}}\\ &= \frac{-\frac{6}{5}}{\frac{4}{3}}\\ &= \frac{-6}{5}\times\frac{3}{4}\\ &= \frac{-18}{20}=-\frac{9}{10}\end{aligned}$$

2. First, draw a diagram. In the diagram below, his house, school and market are represented by H, S, and M, respectively.



Notice that $\angle HSM = 90°$. The angle formed by SM and the north-south line is 45°. The angle formed by HS and the north-south line is also 45°, because it is an alternate interior angle of the 45° bearing of S from H. This gives:

$$\angle HSM = 45° + 45° = 90°$$

ΔHSM is a right-angled triangle. Apply Pythagoras' theorem to find |HM|:

$$\begin{aligned} |HM|^2 &= |HS|^2 + |SM|^2 \\ &= 3^2 + 4^2 \\ &= 9 + 16 \\ &= 25 \\ |HM| &= \sqrt{25} = 5 \text{ km} \end{aligned}$$

Foday is 5 kilometres from his house.

3. a. Apply the tangent ratio to angles 45° and 70°. This will give simultaneous linear equations with 2 unknowns, |QT| and |ST|.

$$\begin{aligned} \tan 45° &= \frac{|ST|}{30+|QT|} \\ 1 &= \frac{|ST|}{30+|QT|} \\ 30 + |QT| &= |ST| ---(1) \end{aligned}$$

$$\begin{aligned} \tan 70° &= \frac{|ST|}{|QT|} \\ 2.75 &= \frac{|ST|}{|QT|} \\ 2.75|QT| &= |ST| ---(2) \end{aligned}$$

Solve the system of equations using substitution:

$$\begin{aligned} 30 + |QT| &= 2.75|QT| \quad \text{Substituting equation (1) into equation (2)} \\ 30 &= 2.75|QT| - |QT| \\ 30 &= 1.75|QT| \\ \frac{30}{1.75} &= |QT| \\ |QT| &= 17.1 \end{aligned}$$

Substitute $|QT|$ into ether linear equation to find $|ST|$:

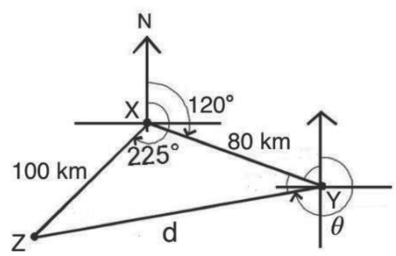
$$|ST| = 30 + |QT| = 30 + 17.1 = 47.1 \text{ m.}$$

b. Apply Pythagoras' theorem to find $|PS|$:

$$\begin{aligned} |PT|^2 + |ST|^2 &= |PS|^2 \\ 47.1^2 + 47.1^2 &= |PS|^2 \\ 4436.82 &= |PS|^2 \\ \sqrt{4436.82} &= |PS| \\ 66.6 \text{ m.} &= |PS| \end{aligned}$$

Lesson Title: Mock Examination: Paper 2B – Advanced Questions	
Practice Activity: PHM4-L094	

1. Draw a diagram:



a. In the triangle, 2 sides and the angle between them are known. The cosine rule can be used. Note that the angle inside the triangle at X is $X = 225° - 120° = 105°$.

$\lvert YZ\rvert^2$	$=$	$\lvert XZ\rvert^2 + \lvert XY\rvert^2 - 2\lvert XZ\rvert\lvert XY\rvert \cos X$	Formula
d^2	$=$	$100^2 + 80^2 - 2(100)(80)\ \cos(105°)$	Substitute values from triangle
	$=$	$10{,}000 + 6{,}400 - 16{,}000 \cos 105°$	
	$=$	$16{,}400 - 16{,}000\ (-0.2588)$	Substitute $\cos 105° = -0.2588$
	$=$	$16{,}400 + 4140.8$	
d^2	$=$	$20{,}540.8$	
d	$=$	$\sqrt{20{,}540.8} = 143$ km to 3 s.f.	Take the square root of both sides

b. To find the bearing of Z from Y, identify the other angles at Y and subtract them from 360°. The other angle outside of the triangle at Y is 60° because it is the alternate interior angle with the 60° angle formed by the bearing and north-south line at point X.

The angle inside the triangle (call it y) can be found using the sine rule:

$\frac{143}{\sin 105°}$	$=$	$\frac{100}{\sin y}$	Substitute in the formula
$\sin y$	$=$	$\frac{100 \sin 105°}{143}$	Solve for y
$\sin y$	$=$	$\frac{100(0.9659)}{143}$	Use the sine table
$\sin y$	$=$	0.6755	
y	$=$	$\sin^{-1}(0.6755)$	
y	$=$	$42.5°$	

Subtract the known angles from 360 to find the bearing:

$$\theta = 360° - 60° - 42.5° = 257.5° = 258° \text{ to 3 s.f.}$$

Bearing: $\overrightarrow{YZ} = (143 \text{ km}, 258°)$

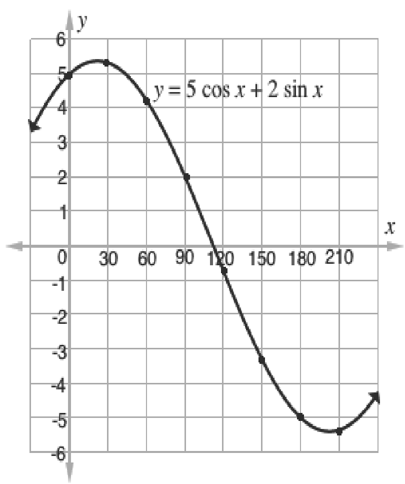
2. a. Completed table (see calculations below):

x	$0°$	$30°$	$60°$	$90°$	$120°$	$150°$	$180°$	$210°$
y	5	5.3	4.2	2	-0.8	-3.3	-5	-5.3

Calculations:

$$\begin{aligned}
5\cos 30° + 2\sin 30° &= 5(0.866) + 2(0.5) \\
&= 5.3 \\
5\cos 90° + 2\sin 90° &= 5(0) + 2(1) \\
&= 2 \\
5\cos 150° + 2\sin 150° &= 5(-0.866) + 2(0.5) \\
&= -3.3 \\
5\cos 180° + 2\sin 180° &= 5(-1) + 2(0) \\
&= -5
\end{aligned}$$

b. Graph (not to scale; ensure that tick marks are 2 cm apart on your graph):



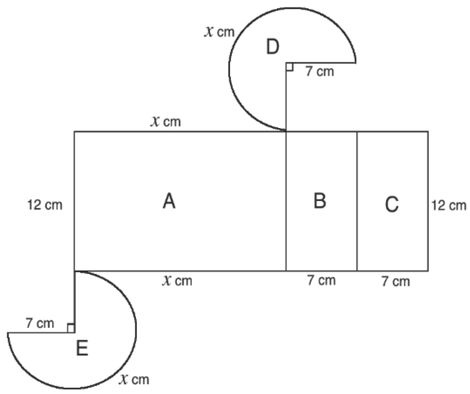
c. The solution to $5\cos x + 2\sin x = 0$ consists of the points where the curve intersects the x-axis. On the interval $0° \leq x \leq 210°$, this only occurs at one point. The solution is approximately $x = 110°$.

d. The maximum value of y is approximately 5.3.

3. a. Since $90°$ was removed, the angle remaining in the solid is $360° - 90° = 270°$. Use $270°$ as a fraction of $360°$ (one full rotation) to calculate the volume.

$$\begin{aligned} V &= \frac{270}{360}\pi r^2 h && \frac{270}{360} \times \text{volume of a cylinder} \\ &= \frac{3}{4}\left(\frac{22}{7}\right)(7^2)(12) \\ &= \frac{3}{4}(22)(7)(12) \\ &= 1386 \text{ cm}^3 \end{aligned}$$

b. To find the surface area, find the area of each of the 5 faces and add them. To facilitate this, draw a net:



Find the measure of unknown length x. Note that it is $\frac{270}{360}$ the circumference of the circle.

$$\begin{aligned} x &= \frac{270}{360}2\pi r && \frac{270}{360} \times \text{circumference of a circle} \\ &= \frac{3}{4}(2)\left(\frac{22}{7}\right)(7) \\ &= \frac{3}{4}(2)(22) \\ &= 33 \text{ cm} \end{aligned}$$

Calculate the area of each shape:

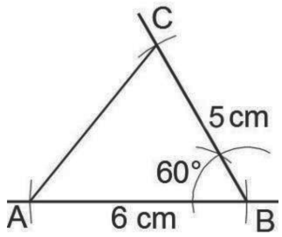
$A = 33 \times 12 = 396$ cm^2

$B = C = 7 \times 12 = 84$ cm^2

$D = E = \frac{270}{360}\pi r^2 = \frac{3}{4}\left(\frac{22}{7}\right)7^2 = \frac{3}{4}(22)7 = 115.5$ cm^2

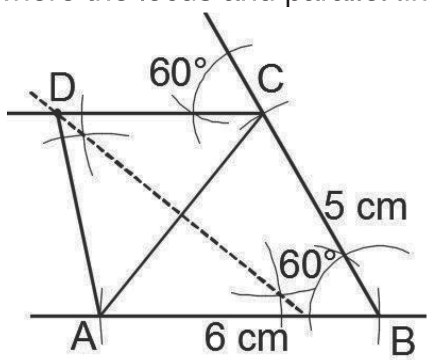
Surface area $= A + B + C + D + E = 396 + 2(84) + 2(115.5) = 795$ cm^2

4. a. Draw the triangle construction, as shown (note that the diagrams are not to scale):



Measure |AC| with a ruler. $|AC| = 5.5$ cm.

b. On the same triangle construction, draw the locus of points equidistant to A and C. Also draw a line from C parallel to |AB|. This can be done in a number of ways. In the diagram below, this is done using a 60° angle at point C. D is the point where the locus and parallel line intersect.



Measure $\angle BAD$ with a protractor. $\angle BAD = 82°$.

Lesson Title: Mock Examination: Paper 2B – Advanced Questions
Practice Activity: PHM4-L095

1. a. Use a table to organize your calculations. Convert each decimal number to a logarithm, and apply the appropriate operations, using BODMAS. Recall that for multiplication of numbers, logarithms are added; for division, they are subtracted. For a square root, the logarithm is divided by 2.

Number	**Logarithm**
20.3	1.3075
$\sqrt{1.568}$	$0.1953 \div 2 = 0.0977$
Product (Numerator)	$1.3075 + 0.0977 = 1.4052$
2.34	0.3692
1.803	0.2560
Product (Denominator)	$0.3692 + 0.2560 = 0.6252$
Division	$1.4052 - 0.6252 = 0.78$

Antilog $0.78 = 6.026$

Answer: $\frac{20.3\times\sqrt{1.568}}{2.34\times1.803} = 6.026$

b. Note that the probability of selecting a Maths book is $\frac{3}{7}$, and the probability of selecting a science book is $\frac{4}{7}$.

i. Multiply to find the probability that both were Maths books:

Pr(Both Maths books)$= \frac{3}{7} \times \frac{3}{7} = \frac{9}{49}$

ii. Multiply to find the probability that one is a Maths book and one is a science book:

Pr(One Maths, one science))$= \frac{3}{7} \times \frac{4}{7} = \frac{12}{49}$

2. a. Organise a cumulative frequency table with upper class boundaries:

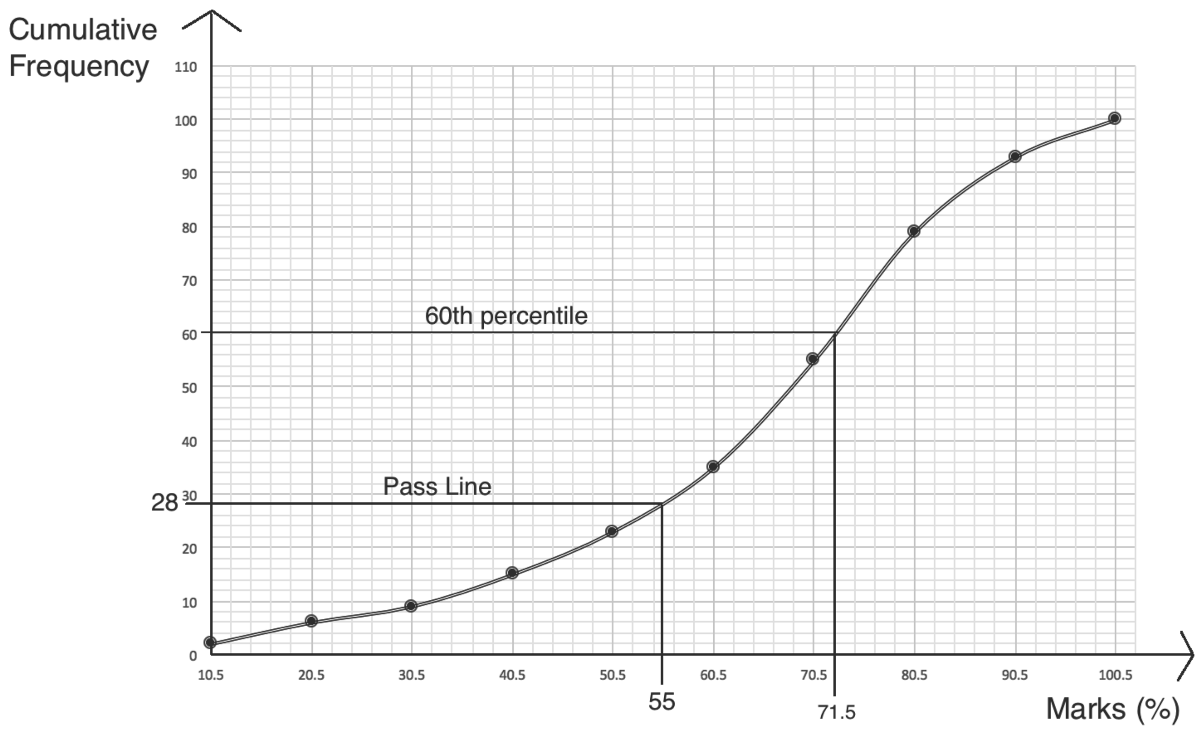
Marks	Upper boundary	Frequency	Cumulative Frequency
1-10	10.5	2	2
11-20	20.5	4	2+4=6
21-30	30.5	3	6+3=9
31-40	40.5	6	9+6=15
41-50	50.5	8	15+8=23
51-60	60.5	12	23+12=35
61-70	70.5	20	35+20=55
71-80	80.5	24	55+24=79
81-90	90.5	14	79+14=93
91-100	100.5	7	93+7=100

Use the table to plot a cumulative frequency curve, with marks on the x-axis and cumulative frequency on the y-axis (see below).

b. i. Find the position of the 60th percentile: $\frac{n}{100}\sum f = \frac{60}{100}(100) = 60$

Identify the 60th percentile using the c.f. curve. Identify 60 on the y-axis – the corresponding x-value gives the percentile. The 60th percentile is 71.5 marks (see curve below).

ii. Find the number of pupils scoring at least 55% using the curve. The corresponding cumulative frequency is 28. Therefore, the number of pupils who passed is $100 - 28 = 72$. Calculate the probability that a random pupil passed: PR(Pupil passed) $= \frac{72}{100} = \frac{18}{25}$.



3.

a. i. Choose 2 sets of x- and y-values from the table. Substitute these into the quadratic equation $y = px^2 + x + q$. This will give simultateous equations that can be solved for p and q.

For example, substitute $(0, -6)$ and $(2, 4)$:

$$\begin{aligned} y &= px^2 + x + q \\ -6 &= p0^2 + 0 + q \\ -6 &= q \end{aligned}$$

$$\begin{aligned} y &= px^2 + x + q \\ 4 &= p2^2 + 2 + q \\ 4 &= 4p + 2 + q \\ 4 - 2 &= 4p + q \\ 2 &= 4p + q \end{aligned}$$

Substitute $q = -6$ into the second equation, and solve for p:

$$\begin{aligned} 2 &= 4p + q \\ 2 &= 4p - 6 \\ 2 + 6 &= 4p \\ 8 &= 4p \\ 2 &= p \end{aligned}$$

We now have $p = 2$ and $q = -6$ the equation $y = 2x^2 + x - 6$.

ii. Complete the table using the function $y = 2x^2 + x - 6$:

x	-4	-3	-2	-1	0	1	2	3
y	22	9	0	-5	-6	-3	4	15

Working:

$$\begin{aligned} y &= 2x^2 + x - 6 \\ &= 2(-4)^2 + (-4) - 6 \\ &= 2(16) - 4 - 6 \\ &= 22 \end{aligned}$$

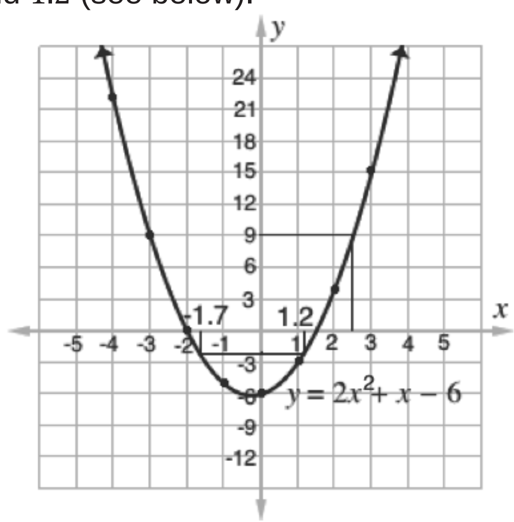
$$\begin{aligned} y &= 2x^2 + x - 6 \\ &= 2(-2)^2 + (-2) - 6 \\ &= 2(4) - 2 - 6 \\ &= 0 \end{aligned}$$

$$\begin{aligned} y &= 2x^2 + x - 6 \\ &= 2(-1)^2 + (-1) - 6 \\ &= 2 - 1 - 6 \\ &= -5 \end{aligned}$$

$$\begin{aligned} y &= 2x^2 + x - 6 \\ &= 2(1)^2 + (1) - 6 \\ &= 2(1) + 1 - 6 \\ &= -3 \end{aligned}$$

b. See the graph below. Note that it is not to scale. Ensure that the tick marks on your x- and y-axes are 2 cm apart.

c. i. Identify $x = 2.5$ on the graph, which corresponds to $y = 9$. (see below).
ii. Identify $y = -2$ on the graph, which has 2 corresponding x-values, approximately -1.7 and 1.2 (see below).



Note that graphs are generally used to make approximations. On the WASSCE exam, examiners accept estimated scores within a certain range. Acceptable answers depend on the scale used. For example, for a 2 cm to 1 unit scale, the disparity allowed is 0.2. Therefore, if the exact values for part c. ii. are -1.7 and 1.2, then examiners should accept scores in the range of -1.9 to -1.5, and 1.0 to 1.4.

4. a. i. Add the expressions from all segments of the Venn diagram. Set them equal to 120, and solve for x.

$$
\begin{aligned}
n(U) = 120 &= 3x - 1 + x + 2 + 3x - 3 + 3x + 5x + 1 + x + 3 + 4x + 1 + 17 \\
120 &= 20x + 20 \\
120 - 20 &= 20x \\
100 &= 20x \\
5 &= x
\end{aligned}
$$

ii. Note that $A \cup B \cup C'$ is the union of A and B, except for those in C. Find the sum of the sections that are in A or B, not including those that are also in C:

$$
\begin{aligned}
n(A \cup B \cup C') &= 3x - 1 + x + 2 + 3x - 3 \\
&= 3(5) - 1 + (5) + 2 + 3(5) - 3 \\
&= 33
\end{aligned}
$$

c. Make x the subject of the equation. Since there is a sine function in the equation, this will require inverse sine to eliminate it.

$4\sin(x + 3.5) - 1$	$=$	0	
$4\sin(x + 3.5)$	$=$	1	Transpose 1
$\sin(x + 3.5)$	$=$	$\frac{1}{4}$	Divide throughout by 4
$\sin(x + 3.5)$	$=$	0.25	Convert to decimal
$x + 3.5$	$=$	$\sin^{-1} 0.25$	Take inverse sine of both sides
$x + 3.5$	$=$	14.5	Substitute $\sin^{-1} 0.25 = 14.48°$
x	$=$	$14.5 - 3.5$	Transpose 3.5
x	$=$	11	

Lesson Title: Mock Examination: Paper 2B – Advanced Questions	
Practice Activity: PHM4-L096	

1. a. Recall the rules for multiplying in modulo. Multiply the 2 numbers of concern, then divide them by the given modulo. The remainder is the answer. For example, consider $4 \otimes 4$. Multiply: $4 \times 4 = 16$. Divide the result by the module, 11: $16 \div 11 = 1r5$. The remainder (5) is written in the table where column 4 and row 4 meet.

   Completed table:

$\otimes$	1	2	4	6	8
1	1	2	4	6	8
2	2	4	8	1	5
4	4	8	5	2	10
6	6	1	2	3	4
8	8	5	10	4	9

   i. To evaluate $(8 \otimes 6) \otimes (4 \otimes 6)$, apply normal order of operation. Remove brackets first.

   $(8 \otimes 6) \otimes (4 \otimes 6) = 4 \otimes 2$ Remove brackets

   $= 8$

   ii. The truth set of $8 \otimes m = 4$ is the set of all m values that make this statement true. There is only one such value in the table, so $m = \{6\}$.

   b. Use the information to create simultaneous equations. Let the numerator of the fraction be x, and the denominator be y. That is, $\frac{x}{y} = \frac{2}{3}$. Consider this equation 1.

   From the problem, we have $2x = y + 12$. Solve this equation for y, we have $y = 2x - 12$. This is equation 2. Substitute equation 2 into equation 1, and solve for x:

$\frac{x}{y}$	=	$\frac{2}{3}$	Equation 1
$\frac{x}{2x-12}$	=	$\frac{2}{3}$	Substitute Equation 2
$3x$	=	$2(2x - 12)$	Cross multiply
$3x$	=	$4x - 24$	Solve for x
x	=	24	

Substitute $x = 24$ into equation 2:

$y = 2x - 12$	Equation 2
$y = 2(24) - 12$	Substitute $x = 24$
$y = 48 - 12$	Solve for y
$y = 36$	

We have $x = 24$ and $y = 36$, which gives the fraction $\frac{24}{36}$

2.

a. Complete the following table to calculate mean deviation. After filling the first 3 columns, calculate mean (shown below) and use it to fill the other columns.

$\boldsymbol{x}$	$\boldsymbol{f}$	$\boldsymbol{fx}$	$\boldsymbol{x - \bar{x}}$	$\boldsymbol{\lvert x - \bar{x} \rvert}$	$\boldsymbol{f\lvert x - \bar{x} \rvert}$
1	8	8	$1 - 3.5 = -2.5$	2.5	20
2	7	14	$2 - 3.5 = -1.5$	1.5	10.5
3	10	30	$3 - 3.5 = -0.5$	0.5	5
4	11	44	$4 - 3.5 = 0.5$	0.5	5.5
5	5	25	$5 - 3.5 = 1.5$	1.5	7.5
6	9	54	$6 - 3.5 = 2.5$	2.5	22.5
Totals:	$\sum f = 50$	$\sum fx = 175$			$\sum f\lvert x - \bar{x} \rvert = 71$

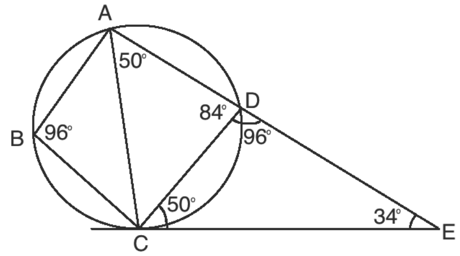
Mean: $\bar{x} = \frac{\sum fx}{\sum f} = \frac{175}{50} = 3.5$ scores

Mean deviation: $\mathrm{MD} = \frac{\sum f\lvert x-\bar{x}\rvert}{\sum f} = \frac{71}{50} = 1.42$

b. Note that this is not the probability of rolling a 3 or higher. It is the probability that, among the rolls in the table, a 3 or higher is selected.

Pr(at least 3) $= \frac{10+11+5+9}{50} = \frac{35}{50} = 0.7$

3. a. Note that $\angle ABC = \angle CDE = 96°$, because $\angle CDE$ is the opposite exterior angle of a cyclic quadrilateral. Also, $\angle CAD = \angle DCE = 50°$, because these are angles in alternate segments. We now have 2 of the 3 angles of triangle CDE. Subtract from $180°$ to find $\angle CED$: $\angle CED = 180° - 96° - 50 = 34°$.



b.

i. Note that, because the triangle is isosceles, $\angle OYZ = \frac{1}{2}\angle XYZ = \frac{1}{2}94° = 47°$.

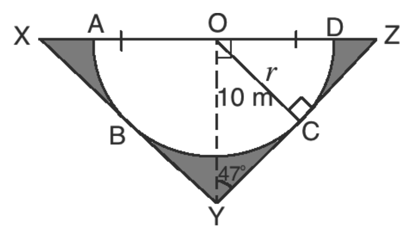
Also note that the radius of the circle OC is perpendicular to the tangent line YZ, according to circle theorems. Therefore, use right-angled triangle OCY to find the radius of the circle, OC:

$$\begin{aligned}
\sin 47° &= \frac{|OC|}{10} \\
0.7314 &= \frac{|OC|}{10} && \text{Substitute } \sin 47° = 0.7314 \text{ from sine table} \\
0.7314 \times 10 &= |OC| \\
7.314 &= |OC|
\end{aligned}$$

Use $r = 7.314$ to find the area of the semicircle:

$$\begin{aligned}
A &= \frac{1}{2}\pi r^2 \\
&= \frac{1}{2}\left(\frac{22}{7}\right)7.314^2 \\
&= 84.1 \text{ m}^2
\end{aligned}$$

ii. To find the area of the shaded portion, subtract the area of the semicircle from the area of the triangle. The height of the triangle is $|OY| = h = 10$ m. Find the base using right-angled triangle YOZ.



$$\begin{aligned}
\tan 47° &= \frac{|OZ|}{10} \\
1.072 &= \frac{|OZ|}{10} && \text{Substitute } \tan 47° = 1.072 \text{ from sine table} \\
1.072 \times 10 &= |OZ| \\
10.72 &= |OZ|
\end{aligned}$$

Base of the triangle = $|OX|+|OZ| = 2|OZ| = 2(10.72) = 21.44$

Area of the triangle:

$$\begin{aligned}
A &= \frac{1}{2}bh \\
&= \frac{1}{2}(21.44)10 \\
&= 107.2 \text{ m}^2
\end{aligned}$$

Area of the shaded portion: $107.2 - 84.1 = 23.1$ m^2

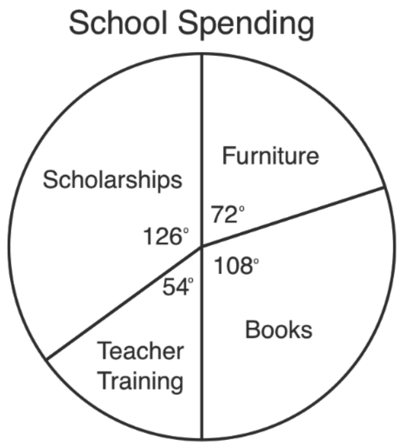
4. a. Write each percentage as a fraction, and use them to find the degree of each sector in the pie chart. Remember that there are 360° in a full rotation:

Furniture: $\frac{20}{100} \times 360° = 72°$

Books: $\frac{30}{100} \times 360° = 108°$

Teacher training: $\frac{15}{100} \times 360° = 54°$

Scholarships: $\frac{35}{100} \times 360° = 126°$



Check your calculations by adding them: $72° + 108° + 54° + 126° = 360°$

Use a protractor and these degree measures to draw a pie chart:

b.

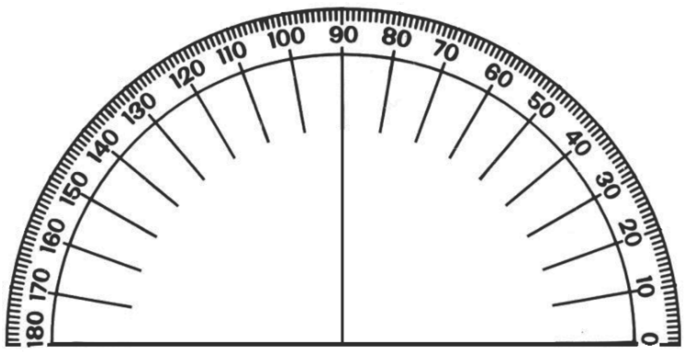
Amount spent on scholarships: $\frac{35}{100} \times 5{,}000 = 1{,}750$

Amount spent on teacher training: $\frac{15}{100} \times 5{,}000 = 750$

Percentage increase $= \frac{1{,}750-750}{750} \times 100 = \frac{1{,}000}{750} \times 100 = 133\%$

Appendix I: Protractor

You can use a protractor to measure angles. If you do not have a protractor, you can make one with paper. Trace this protractor with a pen onto another piece of paper. Then, cut out the semi-circle using scissors.



Appendix II: Sines of Angles

Sines of Angles (x in degrees)

$x \rightarrow \sin x$

											ADD Differences								
x	·0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1	2	3	4	5	6	7	8	9
0	0·0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	2	3	5	7	9	10	12	14	16
1	0·0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	2	3	5	7	9	10	12	14	16
2	0·0349	0366	0384	0401	0419	0436	0454	0471	0488	0506	2	3	5	7	9	10	12	14	16
3	0·0523	0541	0558	0576	0593	0610	0628	0645	0663	0680	2	3	5	7	9	10	12	14	16
4	0·0698	0715	0732	0750	0767	0785	0802	0819	0837	0854	2	3	5	7	9	10	12	14	16
5	0·0872	0889	0906	0924	0941	0958	0976	0993	1011	1028	2	3	5	7	9	10	12	14	16
6	0·1045	1063	1080	1097	1115	1132	1149	1167	1184	1201	2	3	5	7	9	10	12	14	16
7	0·1219	1236	1253	1271	1288	1305	1323	1340	1357	1374	2	3	5	7	9	10	12	14	16
8	0·1392	1409	1426	1444	1461	1478	1495	1513	1530	1547	2	3	5	7	9	10	12	14	16
9	0·1564	1582	1599	1616	1633	1650	1668	1685	1702	1719	2	3	5	7	9	10	12	14	15
10	0·1736	1754	1771	1788	1805	1822	1840	1857	1874	1891	2	3	5	7	9	10	12	14	15
11	0·1908	1925	1942	1959	1977	1994	2011	2028	2045	2062	2	3	5	7	9	10	12	14	15
12	0·2079	2096	2113	2130	2147	2164	2181	2198	2215	2233	2	3	5	7	9	10	12	14	15
13	0·2250	2267	2284	2300	2317	2334	2351	2368	2385	2402	2	3	5	7	8	10	12	14	15
14	0·2419	2436	2453	2470	2487	2504	2521	2538	2554	2571	2	3	5	7	8	10	12	14	15
15	0·2588	2605	2622	2639	2656	2672	2689	2706	2723	2740	2	3	5	7	8	10	12	13	15
16	0·2756	2773	2790	2807	2823	2840	2857	2874	2890	2907	2	3	5	7	8	10	12	13	15
17	0·2924	2940	2957	2974	2990	3007	3024	3040	3057	3074	2	3	5	7	8	10	12	13	15
18	0·3090	3107	3123	3140	3156	3173	3190	3206	3223	3239	2	3	5	7	8	10	12	13	15
19	0·3256	3272	3289	3305	3322	3338	3355	3371	3387	3404	2	3	5	7	8	10	12	13	15
20	0·3420	3437	3453	3469	3486	3502	3518	3535	3551	3567	2	3	5	7	8	10	11	13	15
21	0·3584	3600	3616	3633	3649	3665	3681	3697	3714	3730	2	3	5	6	8	10	11	13	15
22	0·3746	3762	3778	3795	3811	3827	3843	3859	3875	3891	2	3	5	6	8	10	11	13	15
23	0·3907	3923	3939	3955	3971	3987	4003	4019	4035	4051	2	3	5	6	8	10	11	13	14
24	0·4067	4083	4099	4115	4131	4147	4163	4179	4195	4210	2	3	5	6	8	10	11	13	14
25	0·4226	4242	4258	4274	4289	4305	4321	4337	4352	4368	2	3	5	6	8	9	11	13	14
26	0·4384	4399	4415	4431	4446	4462	4478	4493	4509	4524	2	3	5	5	8	9	11	12	14
27	0·4540	4555	4571	4586	4602	4617	4633	4648	4664	4679	2	3	5	6	8	9	11	12	14
28	0·4695	4710	4726	4741	4756	4772	4787	4802	4818	4833	2	3	5	6	8	9	11	12	14
29	0·4848	4863	4879	4894	4909	4924	4939	4955	4970	4985	2	3	5	6	8	9	11	12	14
30	0·5000	5015	5030	5045	5060	5075	5090	5105	5120	5135	2	3	5	6	8	9	11	12	14
31	0·5150	5165	5180	5195	5210	5225	5240	5255	5270	5284	1	3	4	6	7	9	10	12	13
32	0·5299	5314	5329	5344	5358	5373	5388	5402	5417	5432	1	3	4	6	7	9	10	12	13
33	0·5446	5461	5476	5490	5505	5519	5534	5548	5563	5577	1	3	4	6	7	9	10	12	13
34	0·5592	5606	5621	5635	5650	5664	5678	5693	5707	5721	1	3	4	6	7	9	10	12	13
35	0·5736	5750	5764	5779	5793	5807	5821	5835	5850	5864	1	3	4	6	7	9	10	11	13
36	0·5878	5892	5906	5920	5934	5948	5962	5976	5990	6004	1	3	4	6	7	8	10	11	13
37	0·6018	6032	6046	6060	6074	6088	6101	6115	6129	6143	1	3	4	6	7	8	10	11	12
38	0·6157	6170	6184	6198	6211	6225	6239	6252	6266	6280	1	3	4	5	7	8	10	11	12
39	0·6293	6307	6320	6334	6347	6361	6374	6388	6401	6414	1	3	4	5	7	8	9	11	12
40	0·6428	6441	6455	6468	6481	6494	6508	6521	6534	6547	1	3	4	5	7	8	9	11	12
41	0·6561	6574	6587	6600	6613	6626	6639	6652	6665	6678	1	3	4	5	7	8	9	10	12
42	0·6691	6704	6717	6730	6743	6756	6769	6782	6794	6807	1	3	4	5	6	8	9	10	12
43	0·6820	6833	6845	6858	6871	6884	6896	6909	6921	6934	1	3	4	5	6	8	9	10	11
44	0·6947	6959	6972	6984	6997	7009	7022	7034	7046	7059	1	2	4	5	6	7	9	10	11
45	0·7071	7083	7096	7108	7120	7133	7145	7157	7169	7181	1	2	4	5	6	7	9	10	11
46	0·7193	7206	7218	7230	7242	7254	7266	7278	7290	7302	1	2	4	5	6	7	8	10	11
47	0·7314	7325	7337	7349	7361	7373	7385	7396	7408	7420	1	2	4	5	6	7	8	9	11
48	0·7431	7443	7455	7466	7478	7490	7501	7513	7524	7536	1	2	3	5	6	7	8	9	10
49	0·7547	7559	7570	7581	7593	7604	7615	7627	7638	7649	1	2	3	5	6	7	8	9	10
50	0·7660	7672	7683	7694	7705	7716	7727	7738	7749	7760	1	2	3	4	6	7	8	9	10
51	0·7771	7782	7793	7804	7815	7826	7837	7848	7859	7869	1	2	3	4	5	7	8	9	10
52	0·7880	7891	7902	7912	7923	7934	7944	7955	7965	7976	1	2	3	4	5	6	7	8	10
53	0·7986	7997	8007	8018	8028	8039	8049	8059	8070	8080	1	2	3	4	5	6	7	8	9
54	0·8090	8100	8111	8121	8131	8141	8151	8161	8171	8181	1	2	3	4	5	6	7	8	9
55	0·8192	8202	8211	8221	8231	8241	8251	8261	8271	8281	1	2	3	4	5	6	7	8	9
56	0·8290	8300	8310	8320	8329	8339	8348	8358	8368	8377	1	2	3	4	5	6	7	8	9
57	0·8387	8396	8406	8415	8425	8434	8443	8453	8462	8471	1	2	3	4	5	6	7	8	8
58	0·8480	8490	8499	8508	8517	8526	8536	8545	8554	8563	1	2	3	4	5	5	6	7	8
59	0·8572	8581	8590	8599	8607	8616	8625	8634	8643	8652	1	2	3	4	4	5	6	7	8
60	0·8660	8669	8678	8686	8695	8704	8712	8721	8729	8738	1	2	3	3	4	5	6	7	8
61	0·8746	8755	8763	8771	8780	8788	8796	8805	8813	8821	1	2	2	3	4	5	6	7	7
62	0·8829	8838	8846	8854	8862	8870	8878	8886	8894	8902	1	2	2	3	4	5	6	6	7
63	0·8910	8918	8926	8934	8942	8949	8957	8965	8973	8980	1	2	2	3	4	5	5	6	7
64	0·8988	8996	9003	9011	9018	9026	9033	9041	9048	9056	1	2	2	3	4	5	5	6	7
65	0·9063	9070	9078	9085	9092	9100	9107	9114	9121	9128	1	1	2	3	4	4	5	6	7
66	0·9135	9143	9150	9157	9164	9171	9178	9184	9191	9198	1	1	2	3	3	4	5	6	6
67	0·9205	9212	9219	9225	9232	9239	9245	9252	9259	9265	1	1	2	3	3	4	5	5	6
68	0·9272	9278	9285	9291	9298	9304	9311	9317	9323	9330	1	1	2	3	3	4	4	5	6
69	0·9336	9342	9348	9354	9361	9367	9373	9379	9385	9391	1	1	2	2	3	4	4	5	6
70	0·9397	9403	9409	9415	9421	9426	9432	9438	9444	9449	1	1	2	2	3	3	4	5	5
71	0·9455	9461	9466	9472	9478	9483	9489	9494	9500	9505	1	1	2	2	3	3	4	4	5
72	0·9511	9516	9521	9527	9532	9537	9542	9548	9553	9558	1	1	2	2	3	3	4	4	5
73	0·9563	9568	9573	9578	9583	9588	9593	9598	9603	9608	0	1	1	2	2	3	3	4	4
74	0·9613	9617	9622	9627	9632	9636	9641	9646	9650	9655	0	1	1	2	2	3	3	4	4
75	0·9659	9664	9668	9673	9677	9681	9686	9690	9694	9699	0	1	1	2	2	3	3	3	4
76	0·9703	9707	9711	9715	9720	9724	9728	9732	9736	9740	0	1	1	2	2	2	3	3	4
77	0·9744	9748	9751	9755	9759	9763	9767	9770	9774	9778	0	1	1	2	2	2	3	3	3
78	0·9781	9785	9789	9792	9796	9799	9803	9806	9810	9813	0	1	1	1	2	2	2	3	3
79	0·9816	9820	9823	9826	9829	9833	9836	9839	9842	9845	0	1	1	1	2	2	2	3	3
80	0·9848	9851	9854	9857	9860	9863	9866	9869	9871	9874	0	1	1	1	1	2	2	2	3
81	0·9877	9880	9882	9885	9888	9890	9893	9895	9898	9900	0	1	1	1	1	2	2	2	2
82	0·9903	9905	9907	9910	9912	9914	9917	9919	9921	9923	0	0	1	1	1	1	2	2	2
83	0·9925	9928	9930	9932	9934	9936	9938	9940	9942	9943	0	0	1	1	1	1	1	2	2
84	0·9945	9947	9949	9951	9952	9954	9956	9957	9959	9960	0	0	1	1	1	1	1	1	2
85	0·9962	9963	9965	9966	9968	9969	9971	9972	9973	9974	0	0	0	1	1	1	1	1	1
86	0·9976	9977	9978	9979	9980	9981	9982	9983	9984	9985	0	0	0	0	1	1	1	1	1
87	0·9986	9987	9988	9989	9990	9990	9991	9992	9993	9993	0	0	0	0	0	0	1	1	1
88	0·9994	9995	9995	9996	9996	9997	9997	9997	9998	9998									
89	0·9998	9999	9999	9999	9999	1·0000	1·0000	1·0000	1·0000	1·0000									
x	·0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1	2	3	4	5	6	7	8	9

Appendix III: Cosines of Angles

											SUBTRACT Differences								
x	·0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1	2	3	4	5	6	7	8	9
0	1·0000	1·000	1·000	1·000	1·000	1·000	·9999	·9999	·9999	·9999									
1	0·9998	·9998	·9998	·9997	·9997	·9997	·9996	·9996	·9995	·9995									
2	0·9994	·9993	·9993	·9992	·9991	·9990	·9990	·9989	·9988	·9987	0	0	0	0	0	0	1	1	1
3	0·9986	·9985	·9984	·9983	·9982	·9981	·9980	·9979	·9978	·9977	0	0	0	0	1	1	1	1	1
4	0·9976	·9974	·9973	·9972	·9971	·9969	·9968	·9966	·9965	·9963	0	0	0	1	1	1	1	1	1
5	0·9962	·9960	·9959	·9957	·9956	·9954	·9952	·9951	·9949	·9947	0	0	1	1	1	1	1	1	2
6	0·9945	·9943	·9942	·9940	·9938	·9936	·9934	·9932	·9930	·9928	0	0	1	1	1	1	1	2	2
7	0·9925	·9923	·9921	·9919	·9917	·9914	·9912	·9910	·9907	·9905	0	0	1	1	1	1	2	2	2
8	0·9903	·9900	·9898	·9895	·9893	·9890	·9888	·9885	·9882	·9880	0	1	1	1	1	2	2	2	2
9	0·9877	·9874	·9871	·9869	·9866	·9863	·9860	·9857	·9854	·9851	0	1	1	1	1	2	2	2	3
10	0·9848	·9845	·9842	·9839	·9836	·9833	·9829	·9826	·9823	·9820	0	1	1	1	2	2	2	3	3
11	0·9816	·9813	·9810	·9806	·9803	·9799	·9796	·9792	·9789	·9785	0	1	1	1	2	2	2	3	3
12	0·9781	·9778	·9774	·9770	·9767	·9763	·9759	·9755	·9751	·9748	0	1	1	2	2	2	3	3	3
13	0·9744	·9740	·9736	·9732	·9728	·9724	·9720	·9715	·9711	·9707	0	1	1	2	2	2	3	3	4
14	0·9703	·9699	·9694	·9690	·9686	·9681	·9677	·9673	·9668	·9664	0	1	1	2	2	3	3	3	4
15	0·9659	·9655	·9650	·9646	·9641	·9636	·9632	·9627	·9622	·9617	0	1	1	2	2	3	3	4	4
16	0·9613	·9608	·9603	·9598	·9593	·9588	·9583	·9578	·9573	·9568	0	1	1	2	2	3	3	4	4
17	0·9563	·9558	·9553	·9548	·9542	·9537	·9532	·9527	·9521	·9516	1	1	2	2	3	3	4	4	5
18	0·9511	·9505	·9500	·9494	·9489	·9483	·9478	·9472	·9466	·9461	1	1	2	2	3	3	4	4	5
19	0·9455	·9449	·9444	·9438	·9432	·9426	·9421	·9415	·9409	·9403	1	1	2	2	3	3	4	5	5
20	0·9397	·9391	·9385	·9379	·9373	·9367	·9361	·9354	·9348	·9342	1	1	2	2	3	4	4	5	6
21	0·9336	·9330	·9323	·9317	·9311	·9304	·9298	·9291	·9285	·9278	1	1	2	3	3	4	4	5	6
22	0·9272	·9265	·9259	·9252	·9245	·9239	·9232	·9225	·9219	·9212	1	1	2	3	3	4	5	5	6
23	0·9205	·9198	·9191	·9184	·9178	·9171	·9164	·9157	·9150	·9143	1	1	2	3	3	4	5	6	6
24	0·9135	·9128	·9121	·9114	·9107	·9100	·9092	·9085	·9078	·9070	1	1	2	3	4	4	5	6	7
25	0·9063	·9056	·9048	·9041	·9033	·9026	·9018	·9011	·9003	·8996	1	2	2	3	4	5	5	6	7
26	0·8988	·8980	·8973	·8965	·8957	·8949	·8942	·8934	·8926	·8918	1	2	2	3	4	5	5	6	7
27	0·8910	·8902	·8894	·8886	·8878	·8870	·8862	·8854	·8846	·8838	1	2	2	3	4	5	6	6	7
28	0·8829	·8821	·8813	·8805	·8796	·8788	·8780	·8771	·8763	·8755	1	2	2	3	4	5	6	7	7
29	0·8746	·8738	·8729	·8721	·8712	·8704	·8695	·8686	·8678	·8669	1	2	3	3	4	5	6	7	8
30	0·8660	·8652	·8643	·8634	·8625	·8616	·8607	·8599	·8590	·8581	1	2	3	4	4	5	6	7	8
31	0·8572	·8563	·8554	·8545	·8536	·8526	·8517	·8508	·8499	·8490	1	2	3	4	5	5	6	7	8
32	0·8480	·8471	·8462	·8453	·8443	·8434	·8425	·8415	·8406	·8396	1	2	3	4	5	6	7	8	8
33	0·8387	·8377	·8368	·8358	·8348	·8339	·8329	·8320	·8310	·8300	1	2	3	4	5	6	7	8	9
34	0·8290	·8281	·8271	·8261	·8251	·8241	·8231	·8221	·8211	·8202	1	2	3	4	5	6	7	8	9
35	0·8192	·8181	·8171	·8161	·8151	·8141	·8131	·8121	·8111	·8100	1	2	3	4	5	6	7	8	9
36	0·8090	·8080	·8070	·8059	·8049	·8039	·8028	·8018	·8007	·7997	1	2	3	4	5	6	7	8	9
37	0·7986	·7976	·7965	·7955	·7944	·7934	·7923	·7912	·7902	·7891	1	2	3	4	5	6	7	8	10
38	0·7880	·7869	·7859	·7848	·7837	·7826	·7815	·7804	·7793	·7782	1	2	3	4	5	6	7	9	10
39	0·7771	·7760	·7749	·7738	·7727	·7716	·7705	·7694	·7683	·7672	1	2	3	4	6	7	8	9	10
40	0·7660	·7649	·7638	·7627	·7615	·7604	·7593	·7581	·7570	·7559	1	2	3	5	6	7	8	9	10
41	0·7547	·7536	·7524	·7513	·7501	·7490	·7478	·7466	·7455	·7443	1	2	3	5	6	7	8	9	10
42	0·7431	·7420	·7408	·7396	·7385	·7373	·7361	·7349	·7337	·7325	1	2	4	5	6	7	8	9	11
43	0·7314	·7302	·7290	·7278	·7266	·7254	·7242	·7230	·7218	·7206	1	2	4	5	6	7	8	10	11
44	0·7193	·7181	·7169	·7157	·7145	·7133	·7120	·7108	·7096	·7083	1	2	4	5	6	7	9	10	11
x	·0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1	2	3	4	5	6	7	8	9

											SUBTRACT Differences								
x	·0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1	2	3	4	5	6	7	8	9
45	0·7071	·7059	·7046	·7034	·7022	·7009	·6997	·6984	·6972	·6959	1	2	4	5	6	7	9	10	11
46	0·6947	·6934	·6921	·6909	·6896	·6884	·6871	·6858	·6845	·6833	1	3	4	5	6	8	9	10	11
47	0·6820	·6807	·6794	·6782	·6769	·6756	·6743	·6730	·6717	·6704	1	3	4	5	6	8	9	10	12
48	0·6691	·6678	·6665	·6652	·6639	·6626	·6613	·6600	·6587	·6574	1	3	4	5	7	8	9	10	12
49	0·6561	·6547	·6534	·6521	·6508	·6494	·6481	·6468	·6455	·6441	1	3	4	5	7	8	9	11	12
50	0·6428	·6414	·6401	·6388	·6374	·6361	·6347	·6334	·6320	·6307	1	3	4	5	7	8	9	11	12
51	0·6293	·6280	·6266	·6252	·6239	·6225	·6211	·6198	·6184	·6170	1	3	4	5	7	8	10	11	12
52	0·6157	·6143	·6129	·6115	·6101	·6088	·6074	·6060	·6046	·6032	1	3	4	5	7	8	10	11	12
53	0·6018	·6004	·5990	·5976	·5962	·5948	·5934	·5920	·5906	·5892	1	3	4	6	7	8	10	11	13
54	0·5878	·5864	·5850	·5835	·5821	·5807	·5793	·5779	·5764	·5750	1	3	4	6	7	9	10	11	13
55	0·5736	·5721	·5707	·5693	·5678	·5664	·5650	·5635	·5621	·5606	1	3	4	6	7	9	10	12	13
56	0·5592	·5577	·5563	·5548	·5534	·5519	·5505	·5490	·5476	·5461	1	3	4	6	7	9	10	12	13
57	0·5446	·5432	·5417	·5402	·5388	·5373	·5358	·5344	·5329	·5314	1	3	4	6	7	9	10	12	13
58	0·5299	·5284	·5270	·5255	·5240	·5225	·5210	·5195	·5180	·5165	1	3	4	6	7	9	10	12	13
59	0·5150	·5135	·5120	·5105	·5090	·5075	·5060	·5045	·5030	·5015	2	3	5	6	8	9	11	12	14
60	0·5000	·4985	·4970	·4955	·4939	·4924	·4909	·4894	·4879	·4863	2	3	5	6	8	9	11	12	14
61	0·4848	·4833	·4818	·4802	·4787	·4772	·4756	·4741	·4726	·4710	2	3	5	6	8	9	11	12	14
62	0·4695	·4679	·4664	·4648	·4633	·4617	·4602	·4586	·4571	·4555	2	3	5	6	8	9	11	12	14
63	0·4540	·4524	·4509	·4493	·4478	·4462	·4446	·4431	·4415	·4399	2	3	5	6	8	9	11	12	14
64	0·4384	·4368	·4352	·4337	·4321	·4305	·4289	·4274	·4258	·4242	2	3	5	6	8	9	11	13	14
65	0·4226	·4210	·4195	·4179	·4163	·4147	·4131	·4115	·4099	·4083	2	3	5	6	8	10	11	13	14
66	0·4067	·4051	·4035	·4019	·4003	·3987	·3971	·3955	·3939	·3923	2	3	5	6	8	10	11	13	14
67	0·3907	·3891	·3875	·3859	·3843	·3827	·3811	·3795	·3778	·3762	2	3	5	6	8	10	11	13	15
68	0·3746	·3730	·3714	·3697	·3681	·3665	·3649	·3633	·3616	·3600	2	3	5	6	8	10	11	13	15
69	0·3584	·3567	·3551	·3535	·3518	·3502	·3486	·3469	·3453	·3437	2	3	5	6	8	10	11	13	15
70	0·3420	·3404	·3387	·3371	·3355	·3338	·3322	·3305	·3289	·3272	2	3	5	7	8	10	12	13	15
71	0·3256	·3239	·3223	·3206	·3190	·3173	·3156	·3140	·3123	·3107	2	3	5	7	8	10	12	13	15
72	0·3090	·3074	·3057	·3040	·3024	·3007	·2990	·2974	·2957	·2940	2	3	5	7	8	10	12	13	15
73	0·2924	·2907	·2890	·2874	·2857	·2840	·2823	·2807	·2790	·2773	2	3	5	7	8	10	12	13	15
74	0·2756	·2740	·2723	·2706	·2689	·2672	·2656	·2639	·2622	·2605	2	3	5	7	8	10	12	13	15
75	0·2588	·2571	·2554	·2538	·2521	·2504	·2487	·2470	·2453	·2436	2	3	5	7	8	10	12	14	15
76	0·2419	·2402	·2385	·2368	·2351	·2334	·2317	·2300	·2284	·2267	2	3	5	7	8	10	12	14	15
77	0·2250	·2233	·2215	·2198	·2181	·2164	·2147	·2130	·2113	·2096	2	3	5	7	9	10	12	13	15
78	0·2079	·2062	·2045	·2028	·2011	·1994	·1977	·1959	·1942	·1925	2	3	5	7	9	10	12	14	15
79	0·1908	·1891	·1874	·1857	·1840	·1822	·1805	·1788	·1771	·1754	2	3	5	7	9	10	12	14	15
80	0·1736	·1719	·1702	·1685	·1668	·1650	·1633	·1616	·1599	·1582	2	3	5	7	9	10	12	14	15
81	0·1564	·1547	·1530	·1513	·1495	·1478	·1461	·1444	·1426	·1409	2	3	5	7	9	10	12	14	16
82	0·1392	·1374	·1357	·1340	·1323	·1305	·1288	·1271	·1253	·1236	2	3	5	7	9	10	12	14	16
83	0·1219	·1201	·1184	·1167	·1149	·1132	·1115	·1097	·1080	·1063	2	3	5	7	9	10	12	14	16
84	0·1045	·1028	·1011	·0993	·0976	·0958	·0941	·0924	·0906	·0889	2	3	5	7	9	10	12	14	16
85	0·0872	·0854	·0837	·0819	·0802	·0785	·0767	·0750	·0732	·0715	2	3	5	7	9	10	12	14	16
86	0·0698	·0680	·0663	·0645	·0628	·0610	·0593	·0576	·0558	·0541	2	3	5	7	9	10	12	14	16
87	0·0523	·0506	·0488	·0471	·0454	·0436	·0419	·0401	·0384	·0366	2	3	5	7	9	10	12	14	16
88	0·0349	·0332	·0314	·0297	·0279	·0262	·0244	·0227	·0209	·0192	2	3	5	7	9	10	12	14	16
89	0·0175	·0157	·0140	·0122	·0105	·0087	·0070	·0052	·0035	·0017	2	3	5	7	9	10	12	14	16
x	·0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1	2	3	4	5	6	7	8	9

Appendix IV: Tangents of Angles

Tangents of Angles (*x* in degrees)

tan x

x →

x	·0	·1	·2	·3	·4	·5	·6	·7	·8	·9	ADD Differences 1	2	3	4	5	6	7	8	9
0	0·0000	·0017	·0035	·0052	·0070	·0087	·0105	·0122	·0140	·0157	2	3	5	7	9	10	12	14	16
1	0·0175	·0192	·0209	·0227	·0244	·0262	·0279	·0297	·0314	·0332	2	3	5	7	9	10	12	14	16
2	0·0349	·0367	·0384	·0402	·0419	·0437	·0454	·0472	·0489	·0507	2	3	5	7	9	10	12	14	16
3	0·0524	·0542	·0559	·0577	·0594	·0612	·0629	·0647	·0664	·0682	2	4	5	7	9	11	12	14	16
4	0·0699	·0717	·0734	·0752	·0769	·0787	·0805	·0822	·0840	·0857	2	4	5	7	9	11	12	14	16
5	0·0875	·0892	·0910	·0928	·0945	·0963	·0981	·0998	·1016	·1033	2	4	5	7	9	11	12	14	16
6	0·1051	·1069	·1086	·1104	·1122	·1139	·1157	·1175	·1192	·1210	2	4	5	7	9	11	12	14	16
7	0·1228	·1246	·1263	·1281	·1299	·1317	·1334	·1352	·1370	·1388	2	4	5	7	9	11	12	14	16
8	0·1405	·1423	·1441	·1459	·1477	·1495	·1512	·1530	·1548	·1566	2	4	5	7	9	11	12	14	16
9	0·1584	·1602	·1620	·1638	·1655	·1673	·1691	·1709	·1727	·1745	2	4	5	7	9	11	13	14	16
10	0·1763	·1781	·1799	·1817	·1835	·1853	·1871	·1890	·1908	·1926	2	4	5	7	9	11	13	14	16
11	0·1944	·1962	·1980	·1998	·2016	·2035	·2053	·2071	·2089	·2107	2	4	5	7	9	11	13	15	16
12	0·2126	·2144	·2162	·2180	·2199	·2217	·2235	·2254	·2272	·2290	2	4	5	7	9	11	13	15	16
13	0·2309	·2327	·2345	·2364	·2382	·2401	·2419	·2438	·2456	·2475	2	4	6	7	9	11	13	15	17
14	0·2493	·2512	·2530	·2549	·2568	·2586	·2605	·2623	·2642	·2661	2	4	5	7	9	11	13	15	17
15	0·2679	·2698	·2717	·2736	·2754	·2773	·2792	·2811	·2830	·2849	2	4	6	8	9	11	13	15	17
16	0·2867	·2886	·2905	·2924	·2943	·2962	·2981	·3000	·3019	·3038	2	4	6	8	9	11	13	15	17
17	0·3057	·3076	·3096	·3115	·3134	·3153	·3172	·3191	·3211	·3230	2	4	6	8	10	12	13	15	17
18	0·3249	·3269	·3288	·3307	·3327	·3346	·3365	·3385	·3404	·3424	2	4	6	8	10	12	14	16	17
19	0·3443	·3463	·3482	·3502	·3522	·3541	·3561	·3581	·3600	·3620	2	4	6	8	10	12	14	16	18
20	0·3640	·3659	·3679	·3699	·3719	·3739	·3759	·3779	·3799	·3819	2	4	6	8	10	12	14	16	18
21	0·3839	·3859	·3879	·3899	·3919	·3939	·3959	·3979	·4000	·4020	2	4	6	8	10	12	14	16	18
22	0·4040	·4061	·4081	·4101	·4122	·4142	·4163	·4183	·4204	·4224	2	4	6	8	10	12	14	16	18
23	0·4245	·4265	·4286	·4307	·4327	·4348	·4369	·4390	·4411	·4431	2	4	6	8	10	12	15	17	19
24	0·4452	·4473	·4494	·4515	·4536	·4557	·4578	·4599	·4621	·4642	2	4	6	8	11	13	15	17	19
25	0·4663	·4684	·4706	·4727	·4748	·4770	·4791	·4813	·4834	·4856	2	4	6	9	11	13	15	17	19
26	0·4877	·4899	·4921	·4942	·4964	·4986	·5008	·5029	·5051	·5073	2	4	7	9	11	13	15	17	20
27	0·5095	·5117	·5139	·5161	·5184	·5206	·5228	·5250	·5272	·5295	2	4	7	9	11	13	16	18	20
28	0·5317	·5340	·5362	·5384	·5407	·5430	·5452	·5475	·5498	·5520	2	5	7	9	11	14	16	18	20
29	0·5543	·5566	·5589	·5612	·5635	·5658	·5681	·5704	·5727	·5750	2	5	7	9	12	14	16	18	21
30	0·5774	·5797	·5820	·5844	·5867	·5890	·5914	·5938	·5961	·5985	2	5	7	9	12	14	16	19	21
31	0·6009	·6032	·6056	·6080	·6104	·6128	·6152	·6176	·6200	·6224	2	5	7	10	12	14	17	19	22
32	0·6249	·6273	·6297	·6322	·6346	·6371	·6395	·6420	·6445	·6469	2	5	7	10	12	15	17	20	22
33	0·6494	·6519	·6544	·6569	·6594	·6619	·6644	·6669	·6694	·6720	3	5	8	10	13	15	18	20	23
34	0·6745	·6771	·6796	·6822	·6847	·6873	·6899	·6924	·6950	·6976	3	5	8	10	13	15	18	21	23
35	0·7002	·7028	·7054	·7080	·7107	·7133	·7159	·7186	·7212	·7239	3	5	8	11	13	16	18	21	24
36	0·7265	·7292	·7319	·7346	·7373	·7400	·7427	·7454	·7481	·7508	3	5	8	11	14	16	19	22	24
37	0·7536	·7563	·7590	·7618	·7646	·7673	·7701	·7729	·7757	·7785	3	6	8	11	14	17	19	22	25
38	0·7813	·7841	·7869	·7898	·7926	·7954	·7983	·8012	·8040	·8069	3	6	9	11	14	17	20	23	26
39	0·8098	·8127	·8156	·8185	·8214	·8243	·8273	·8302	·8332	·8361	3	6	9	12	15	18	21	23	26
40	0·8391	·8421	·8451	·8481	·8511	·8541	·8571	·8601	·8632	·8662	3	6	9	12	15	18	21	24	27
41	0·8693	·8724	·8754	·8785	·8816	·8847	·8878	·8910	·8941	·8972	3	6	9	12	16	19	22	25	28
42	0·9004	·9036	·9067	·9099	·9131	·9163	·9195	·9228	·9260	·9293	3	6	10	13	16	19	22	26	29
43	0·9325	·9358	·9391	·9424	·9457	·9490	·9523	·9556	·9590	·9623	3	7	10	13	17	20	23	27	30
44	0·9657	·9091	·9725	·9759	·9793	·9827	·9861	·9896	·9930	·9965	3	7	10	14	17	21	24	27	31
x	·0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1	2	3	4	5	6	7	8	9

x	·0	·1	·2	·3	·4	·5	·6	·7	·8	·9	ADD Differences 1	2	3	4	5	6	7	8	9
45	1·000	1·003	1·007	1·011	1·014	1·018	1·021	1·025	1·028	1·032	0	1	1	1	2	2	2	3	3
46	1·036	1·039	1·043	1·046	1·050	1·054	1·057	1·061	1·065	1·069	0	1	1	1	2	2	3	3	3
47	1·072	1·076	1·080	1·084	1·087	1·091	1·095	1·099	1·103	1·107	0	1	1	2	2	2	3	3	3
48	1·111	1·115	1·118	1·122	1·126	1·130	1·134	1·138	1·142	1·145	0	1	1	2	2	2	3	3	4
49	1·150	1·154	1·159	1·163	1·167	1·171	1·175	1·179	1·183	1·188	0	1	1	2	2	2	3	3	4
50	1·192	1·196	1·200	1·205	1·209	1·213	1·217	1·222	1·226	1·230	0	1	1	2	2	3	3	4	4
51	1·235	1·239	1·244	1·248	1·253	1·257	1·262	1·266	1·271	1·275	0	1	1	2	2	3	3	4	4
52	1·280	1·285	1·289	1·294	1·299	1·303	1·308	1·313	1·317	1·322	0	1	1	2	2	3	3	4	4
53	1·327	1·332	1·337	1·342	1·347	1·351	1·356	1·361	1·366	1·371	0	1	1	2	2	3	3	4	4
54	1·376	1·381	1·387	1·392	1·397	1·402	1·407	1·412	1·418	1·423	1	1	2	2	3	3	4	4	5
55	1·428	1·433	1·439	1·444	1·450	1·455	1·460	1·466	1·471	1·477	1	1	2	2	3	3	4	4	5
56	1·483	1·488	1·494	1·499	1·505	1·511	1·517	1·522	1·528	1·534	1	1	2	2	3	3	4	5	5
57	1·540	1·546	1·552	1·558	1·564	1·570	1·576	1·582	1·588	1·594	1	1	2	2	3	4	4	5	5
58	1·600	1·607	1·613	1·619	1·625	1·632	1·638	1·645	1·651	1·658	1	1	2	3	3	4	4	5	6
59	1·664	1·671	1·678	1·684	1·691	1·698	1·704	1·711	1·718	1·725	1	1	2	3	3	4	5	5	6
60	1·732	1·739	1·746	1·753	1·760	1·767	1·775	1·782	1·789	1·797	1	1	2	3	4	4	5	6	6
61	1·804	1·811	1·819	1·827	1·834	1·842	1·849	1·857	1·865	1·873	1	2	2	3	4	5	5	6	7
62	1·881	1·889	1·897	1·905	1·913	1·921	1·929	1·937	1·946	1·954	1	2	2	3	4	5	6	7	7
63	1·963	1·971	1·980	1·988	1·997	2·006	2·014	2·023	2·032	2·041	1	2	3	4	4	5	6	7	8
64	2·050	2·059	2·069	2·078	2·087	2·097	2·106	2·116	2·125	2·135	1	2	3	4	5	6	7	8	8
65	2·145	2·154	2·164	2·174	2·184	2·194	2·204	2·215	2·225	2·236	1	2	3	4	5	6	7	8	9
66	2·246	2·257	2·267	2·278	2·289	2·300	2·311	2·322	2·333	2·344	1	2	3	4	5	7	8	9	10
67	2·356	2·367	2·379	2·391	2·402	2·414	2·426	2·438	2·450	2·463	1	2	4	5	6	7	8	10	11
68	2·475	2·488	2·500	2·513	2·526	2·539	2·552	2·565	2·578	2·592	1	3	4	5	7	8	9	10	12
69	2·605	2·619	2·633	2·646	2·660	2·675	2·689	2·703	2·718	2·733	1	3	4	6	7	9	10	11	13
70	2·747	2·762	2·778	2·793	2·808	2·824	2·840	2·856	2·872	2·888	2	3	5	6	8	9	11	13	14
71	2·904	2·921	2·937	2·954	2·971	2·989	3·006	3·024	3·042	3·060	2	3	5	7	9	10	12	14	16
72	3·078	3·096	3·115	3·133	3·152	3·172	3·191	3·211	3·230	3·251	2	4	6	8	10	12	14	15	17
73	3·271	3·291	3·312	3·333	3·354	3·376	3·398	3·420	3·442	3·465	2	4	6	9	11	13	15	17	19
74	3·487	3·511	3·534	3·558	3·582	3·606	3·630	3·655	3·681	3·706	2	5	7	10	12	15	17	20	22
75	3·732	3·758	3·785	3·812	3·839	3·867	3·895	3·923	3·952	3·981	3	6	8	11	14	17	20	22	25
76	4·011	4·041	4·071	4·102	4·134	4·165	4·198	4·230	4·264	4·297	3	6	10	13	16	19	22	26	29
77	4·331	4·366	4·402	4·437	4·474	4·511	4·548	4·586	4·625	4·666									
78	4·705	4·745	4·787	4·829	4·872	4·915	4·959	5·005	5·050	5·097									
79	5·145	5·193	5·242	5·292	5·343	5·396	5·449	5·503	5·558	5·614									
80	5·671	5·730	5·789	5·850	5·912	5·976	6·041	6·107	6·174	6·243	differences unreliable:								
81	6·314	6·386	6·460	6·535	6·612	6·691	6·772	6·855	6·940	7·026									
82	7·115	7·207	7·300	7·396	7·495	7·596	7·700	7·806	7·916	8·028									
83	8·144	8·264	8·386	8·513	8·643	8·777	8·915	9·058	9·205	9·357									
84	9·514	9·677	9·845	10·02	10·20	10·39	10·58	10·78	10·99	11·20									
85	11·43	11·66	11·91	12·16	12·43	12·71	13·00	13·30	13·62	13·95									
86	14·30	14·67	15·06	15·46	15·89	16·35	16·83	17·34	17·89	18·46									
87	19·08	19·74	20·45	21·20	22·02	22·90	23·86	24·90	26·03	27·27									
88	28·64	30·14	31·82	33·69	35·80	38·19	40·92	44·07	47·74	52·08									
89	57·29	63·66	71·62	81·85	95·46	114·6	143·2	191·0	286·5	573·0									
x	·0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1	2	3	4	5	6	7	8	9

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